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President: Prof. F. A. F. C. WENT

Secretary: Prof. B. BROUWER

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CONTENTS

Mathematics. — *Ueber unendlich-dimensionale Punktmengen.* By Dr. W. HUREWICZ. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of May 26, 1928).

Bekanntlich sind für jede natürliche Zahl n in der Gesamtheit aller separablen Räume die höchstens n-dimensionalen dadurch charakterisiert, dass sie sich als Summen von $n+1$ nulldimensionalen Teilen darstellen lassen [1]).

Diese Tatsache legt den Gedanken nahe, die Theorie der unendlich-dimensionalen Mengen mit der Betrachtung jener Räume zu beginnen, die in *abzählbar viele* nulldimensionale Teile (oder, was dasselbe bedeutet, in abzählbar viele endlichdimensionale Teile) zerlegbar sind. Wir wollen solche Räume *abzählbar-dimensional* nennen, wobei aus formalen Gründen die leere Menge ebenfalls als abzählbardimensional bezeichnet wird. Ein abzählbar-dimensionaler Raum, der keine endliche Dimension besitzt heisst $\aleph_0$-dimensional.

Im folgenden soll die Theorie der abzählbar-dimensionalen Mengen in ihren Grundzügen entwickelt werden. Dabei beschränken wir unsere Betrachtungen ausschliesslich auf metrisierbare separable Räume.

1. Vor allem erhebt sich die Frage, ob nicht *jeder* separable Raum abzählbar-dimensional ist [2]). Da nach URYSOHN jede separable Menge mit einer Teilmenge des Hilbertschen unendlich-dimensionalen Raumes H homöomorph ist [3]), so ist unser Problem gleichbedeutend mit der Frage, ob der Hilbertsche Raum abzählbar-dimensional ist.

Die Antwort fällt, wie wir gleich sehen werden, *negativ* aus. *Der Hilbertsche unendlich-dimensionale Raum kann nicht mit abzählbar vielen endlich-dimensionalen Teilen ausgefüllt werden.*

Der Bequemlichkeit halber führen wir den Beweis nicht für den Raum H selbst, sondern (was offenbar auf dasselbe hinauskommt) für den *kompakten* Teil K von H, welcher durch die Koordinatenungleichungen

$$x_n \leqq 1/n$$

definiert ist. Sei also in K irgendwie eine Folge von nulldimensionalen Mengen

$$N_1, N_2, N_3, \ldots$$

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[1]) Vgl. meine Arbeit „Normalbereiche und Dimensionstheorie", Math. **96**, S. 761.

[2]) Vgl. URYSOHN, Math. Ann. **92**, S. 303.

[3]) Dieses Problem wurde bereits von URYSOHN formuliert, vgl. Fund. Math. **8**, S. 351, „Problème ν". Daselbst wurde die Vermutung ausgesprochen, dass Antwort auf das Problem negativ ausfällt.

vorgegeben. Wir haben zu zeigen, dass es in K Punkte gibt, die keiner dieser Mengen angehören.

Bezeichnen wir für jedes natürliche n mit B_n bzw. mit $B^{(n)}$ die Menge aller Punkte von K, für deren n-te Koordinate

$$x_n = 0$$

bzw.

$$x_n = 1/n$$

gilt. Die beiden Mengen B_n und $B^{(n)}$ sind abgeschlossen und zueinander fremd.

Stellen wir jetzt den Raum K als Summe von zwei abgeschlossenen Mengen K_1 und $K^{(1)}$ dar, von denen die erste mit $B^{(1)}$, die zweite mit B_1 punktfremd ist, und so dass der Durchschnitt $K_1 . K^{(1)}$ zu der nulldimensionalen Menge N_1 fremd ist (dass eine solche Zerlegung von K existiert, folgt aus bekannten Sätzen über nulldimensionale Mengen [4])).

Jetzt wird die Menge $K^{(1)}$ analog wie vorher der ganze Raum K, in zwei abgeschlossene Teile K_2 und $K^{(2)}$ zerlegt mit den Eigenschaften:

$$K_2 . B^{(2)} = K^{(2)} . B_2 = 0$$

$$K_2 . K^{(2)} . N_2 = 0.$$

Das Verfahren wird in dieser Weise fortgesetzt. Allgemein haben wir:

$$K^{(m)} = K_{m+1} + K^{(m+1)} \quad \ldots \ldots \ldots \quad (1)$$

$$K_m . B^{(m+1)} = K^{(m)} . B_{m+1} = 0 \quad \ldots \ldots \quad (2)$$

$$K_m . K^{(m)} . N_m = 0 \quad \ldots \ldots \ldots \quad (3)$$

Sei jetzt m eine natürliche Zahl. Betrachten wir das in K liegende m-dimensionale Intervall

$$x_i \leqq 1/i \, (i = 1, 2 \ldots m) \quad ; \quad x_i = 0 \, (i = m+1, m+2, \ldots). \quad . \quad (')$$

und das $(m+1)$-tupel der abgeschlossenen Mengen:

$$K_1, K_2, \ldots, K_m, K^{(m)}, \ldots \ldots \ldots \quad ('')$$

die in ihrer Gesamtheit den ganzen Raum K ausfüllen.

Unter Berücksichtigung der Beziehungen (2) ergibt sich aus einem grundlegenden Lebesgue-Brouwerschen Satze [5]), dass es innerhalb (') Punkte gibt, die sämtlichen $n+1$ Mengen ('') angehören. Bezeichnen wir mit P_m den Durchschnitt der Mengen (''), so sind die Mengen P_1, $P_2, P_3, \ldots$ nicht-leer, abgeschlossen, und jede jede von ihnen ist in der

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[4]) Vgl. die sub [1]) zitierte Abhandlung, S. 739, Satz III.

[5]) Diesen Satz kann man in der folgenden Form aussprechen: Werden mit $B_1, B_2 \ldots B_n$, $B_1', B_2' \ldots B_n'$ die $2n$ Seiten des n-dimensionalen Intervalls I bezeichnet, wobei B_i und B_i' gegenüberliegende Seiten bedeuten, und wird I in $n+1$ abgeschlossene Mengen $A_1, A_2 \ldots A_{n+1}$ zerlegt, so dass $A_i . B_i = 0 \; (1 = 1, 2 \ldots n)$ und $A_i . B_k = 0$ für $i > k$ gilt, so haben die $n+1$ Mengen A_i mindestens einen gemeinsamen Punkt. Vgl. LEBESGUE, Math. Ann. **70**, S. 166—168 und Fund. Math. 2, S. 257; BROUWER, Crellscher Journal **142**, **150**, wo der Satz zum ersten Mal bewiesen wurde. — Siehe ferner HUREWICZ, Math. Ann., wo sich die hier angeführte Form des Satzes findet.

vorangehenden enthalten. Nach dem sogen. Cantorschen Durchschnittsatz ist der Durchschnitt $\prod_{i=1}^{\infty} P_i$ nicht leer, anderseits ist dieser Durchschnitt zu sämtlichen N_i fremd, denn wegen (3) ist P_i zu N_i fremd. Es sind also in K Punkte vorhanden, die keinem N_i angehören, und damit ist gezeigt, dass K nicht abzählbar-dimensional ist.

2. Das einfachste Beispiel einer $\aleph_0$-dimensionalen Menge bildet die Gesamtheit aller Punkte des Hilbertschen Raumes H, die nur endlich viele von Null verschiedene Koordinaten haben. Diese Menge ist allerdings nicht kompakt.

Um ein Beispiel eines kompakten $\aleph_0$-dimensionalen Raumes zu bilden, bezeichnen wir mit W_m $(m = 1, 2, \ldots)$ die Menge aller Punkte von H, die der Bedingung

$$x_i \leqq 1/m \,(i = 1, 2 \ldots m) \quad x_i = 0 \,(i = m+1, m+2, \ldots)$$

genügen. Die Summe der endlich-dimensionalen Mengen W_m ist $\aleph_0$-dimensional und, wie man leicht sieht, kompakt.

Durch ein komplizierteres Verfahren könnte man einen kompakten abzählbar-dimensionalen Raum bilden, in dem jede offene Menge unendlich-dimensional ist (der im vorigen Beispiel definierte kompakte Raum hat offenbar diese Eigenschaft nicht). Ein derartiger Raum kann, wie sich leicht beweisen liesse, nicht als Summe von abzählbar vielen in ihm *abgeschlossenen* endlich-dimensionalen Mengen dargestellt werden.

3. Die Überlegungen dieses und des folgenden Paragraphen werden die Ausnahmestellung zeigen, welche die abzählbar-dimensionalen Mengen in Bezug auf die übrigen Punktmengen einnehmen.

Wir gehen von dem folgenden Satze aus:

Hat ein separabler Raum die Eigenschaft, dass zu jedem seiner Punkte beliebig kleine Umgebungen existieren, deren Begrenzungen abzählbar-dimensional sind, so ist der Raum R selbst abzählbar-dimensional.

Durch Anwendung eines allgemeinen Satzes von mir [6]) ergibt sich nämlich zunächst, dass R in zwei Mengen zerlegbar ist, von denen die eine nulldimensional, die zweite abzählbar-dimensional ist; daraus folgt aber sofort die Behauptung.

Versuchen wir also die Rekursionsprinzipien von BROUWER, MENGER und URYSOHN, welche den Übergang von der Klasse der n-dimensionalen Räume zu der Klasse der $(n+1)$-dimensionalen Räume vermitteln, auf den Bereich der abzählbar-dimensionalen Räume als Ausgangsbereich anzuwenden, so gelangen wir auf diesem Wege, wie der soeben angeführte Satz zeigt, zu keiner neuen Klasse von Räumen.

Wir führen jetzt die folgende Begriffsbildung ein: Ein System $\mathfrak{A}$ von

[6]) Nämlich des Theorems I in der sub [1]) zitierten Arbeit, S.754.

separablen Räumen nennen wir *komplett* im System $\mathfrak{B}$ aller separablen Räume, wenn die beiden folgenden Bedingungen erfüllt sind: 1. der "leere Raum" (die „leere Menge") kommt im System $\mathfrak{A}$ vor. 2. Besitzt der separable Raum R zu jedem seiner Punkte beliebig kleine Umgebungen, deren Begrenzungen Räume aus dem Bereich $\mathfrak{A}$ sind, so gehört R ebenfalls dem System $\mathfrak{A}$ an.

Indem wir statt des Systems aller separablen Räume etwa das weniger umfangreiche System $\mathfrak{K}$ aller kompakten Räume als Operationsbereich zugrundelegen, definieren wir in analoger Weise, was unter einem in Bezug auf $\mathfrak{K}$ kompletten Bereich von kompakten Räumen zu verstehen ist.

Das oben ausgesprochene Theorem können wir jetzt auch so formulieren. Im Bereiche aller separablen Räume bilden die abzählbar-dimensionalen Räume einen kompletten Bereich.

Es ist nun sehr merkwürdig, dass unter Beschränkung auf kompakte Räume gewissermassen auch die Umkehrung dieses Satzes gilt: *Ein kompakter abzählbar-dimensionaler Raum gehört jedem beliebigen kompletten Bereich an.*

Angenommen nämlich, der kompakte abzählbare-dimensionale Raum

$$R = \sum_{i=1}^{\infty} N_i \qquad (\dim N_i = 0;\ i = 1, 2, \ldots)$$

komme in einem kompletten Bereich $\mathfrak{A}$ nicht vor. Dann muss es nach der Definition der kompletten Bereiche einen Punkt p von R und eine Umgebung U von p geben, so dass keine in U enthaltene Umgebung von p durch eine dem Bereich $\mathfrak{A}$ angehörende Menge begrenzt ist.

Wählen wir innerhalb U eine Umgebung von p, deren Begrenzung B_1 zu der nulldimensionalen Menge N_1 fremd ist. Die abgeschlossene Menge B_1 ist kein Element von $\mathfrak{A}$, woraus folgt, dass diese Menge nicht leer ist. Indem wir jetzt mit B_1 in derselben Weise verfahren, wie soeben mit R, definieren wir eine abgeschlossene Teilmenge B_2 von B_1, die in $\mathfrak{A}$ nicht vorkommt und mit N_2 punktfremd ist.

Das Verfahren ad infinitum fortsetzend, erhalten wir eine absteigende Folge von abgeschlossenen, nicht leeren Mengen $B_1, B_2, B_3, \ldots,$ die k^{te} von denen zu N_k fremd ist $(k = 1, 2, \ldots)$. Der Durchschnitt der B_i ist zu sämtlichen N_k fremd, also leer, was einen Widerspruch mit dem Cantorschen Durchschnittsatz ergibt.

Die beiden oben bewiesenen Sätze können wir in dem folgenden *Fundamentalsatze* zusammenfassen:

Innerhalb des Bereiches $\mathfrak{K}$ aller kompakten Räume bilden die abzählbar-dimensionalen Räume den kleinsten kompletten Bereich.

Dieses Theorem gibt eine von der Zerlegbarkeit in nulldimensionale Teile unabhängige Definition der abzählbar-dimensionalen kompakten Räume.

Der Fundamentalsatz liefert eine Methode um allgemeine Sätze über abzählbar-dimensionale Räume zu beweisen: Sobald man weiss, dass die

Räume denen eine gegebene Eigenschaft E zukommt, einen kompletten Bereich bilden, kann man daraus schliessen, dass diese Eigenschaft *jedem* kompakten abzählbar-dimensionalen Raum zukommt.

Mit Hilfe dieser Methode kann man beispielweise zeigen, dass ein kompakter $\aleph_0$-dimensionaler Raum Teilmengen (sogar abgeschlossene) von beliebig vorgeschriebener endlicher Dimension enthält. [7]) Dies folgt einfach aus der Bemerkung, dass für jedes natürliche n alle diejenigen Räume, welche entweder weniger als n-dimensional sind, oder einen genau n-dimensionalen abgeschlossenen Teil enthalten, einen kompletten Bereich bilden.

4. Wir gehen jetzt von den kompakten Räumen zu den allgemeinen separablen Räumen über und fragen, ob auch in dem Bereich der separablen Räume der Satz gilt, dass die abzählbar-dimensionalen Räume den kleinsten kompletten Bereich bilden. Es zeigt sich, dass dies nicht der Fall ist. Es gilt vielmehr die folgende Behauptung, auf deren Beweis wir hier nicht eingehen können.

Innerhalb des Bereiches aller separabler Räume wird der kleinste komplette Bereich von allen jenen Räumen gebildet, die nicht nur selbst abzählbar-dimensional sind, sondern überdies noch mit einer Teilmenge eines kompakten *abzählbar-dimensionalen Raumes homöomorph sind.* [8])

Auf der anderen Seite zeigt es sich, dass es abzählbar-dimensionale separable Räume gibt, die sich nicht in einen kompakten abzählbar-dimensionalen Raum einbetten lassen (in dem Sinne, dass sie mit keiner Teilmenge eines derartigen Raumes homöomorph sind). Zu diesen Räumen gehört beispielweise der bereits oben betrachtete Raum, der von jenen Punkten des Hilbertschen Raumes gebildet wird, die nur endlich viele nicht verschwindende Koordinaten besitzen. Dies ergibt einen scharfen Gegensatz mit der Theorie der endlichdimensionalen Räume, wo doch der Satz gilt, dass jede endlich-dimensionale Menge in einen kompakten Raum von derselben Dimension einbettbar ist. [9])

In diesem Zusammenhang sei noch bemerkt, dass jeder abzählbar-dimensionale Raum, der vollständig und separabel ist, sich auf eine Teilmenge eines kompakten abzählbar-dimensionalen Raumes topologisch abbilden lässt, also dem kleinsten kompletten Bereich angehört. Der am

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[7]) Dies hängt mit der mir von MENGER mündlich mitgeteilten Frage zusammen, ob jeder unendlich-dimensionale Raum Teilmengen von jeder endlichen Dimension enthält. Es ist mir gelungen zu zeigen, dass bei der Annahme der Kontinuumshypothese die Mengersche Frage, wenn man sie in voller Allgemeinheit ausspricht, negativ beantwortet werden muss. Unter Annahme der Kontinuumshypothese lassen sich nämlich Räume angeben, die nur abzählbare und unendlich-dimensionale Teilmengen enthalten.

[8]) Recht schwierig ist der Nachweis, dass die erwähnten Räume tatsächlich einen kompletten Bereich bilden. Dass dies der kleinste komplette Bereich ist, beweist man hingegen beinahe ebenso einfach, wie oben im Falle kompakter Räume.

[9]) Siehe HUREWICZ, diese Proceedings, **30** (1927), S. 425—430.

Schluss des letzten Paragraphen ausgesprochene Satz für kompakte Räume ist somit auch für separable vollständige Raume gültig.

5. Die Überlegungen der letzten Paragraphen rücken in ein neues Licht, wenn wir an den Begriff der transfiniten Zahl anknüpfen.

Unserer Definition der abzählbar-dimensionalen Mengen lag eine Auffassung der Dimensionszahl als *Kardinalzahl* zugrunde. Diese Kardinalzahl wird bestimmt durch die kleinste Anzahl von nulldimensionalen Teilen, welche zum Aufbau der gegebenen Menge nötig ist. Die Berechtigung dieser Auffassung ergab sich aus dem am Anfang dieser Arbeit angeführten "Zerlegungstheorem" für endlich dimensionale Mengen.

Anderseits führt uns aber die Verallgemeinerung der üblichen rekursiven Definition der Dimensionszahl darauf, gewissen unendlich-dimensionalen Mengen zur Charakterisierung ihrer Dimensionsverhältnisse eine transfinite *Ordinalzahl* zuzuordnen, welche wir als die "Ordinaldimension" der betreffenden Menge bezeichnen wollen. An die Mengersche Form der Dimensionsdefinition anknüpfend, setzen wir nämlich fest, dass für eine Ordinalzahl α unter einem Raum von der Ordinalmension α ein Raum zu verstehen ist, der erstens keine kleinere Ordinaldimension besitzt und der zweitens zu jedem seiner Punkte beliebig kleine Umgebungen enthält, deren Begrenzungen Ordinaldimensionen $< \alpha$ haben.

Vor allem bemerken wir, dass einem *separablen* Raum auf Grund dieser Definition, wenn überhaupt, dann stets eine *abzählbare* Ordinalzahl (Ordinalzahl der ersten oder zweiten Zahlenklasse) als Dimension zugeordnet wird.

Ferner ist es klar, dass unsere Definition keineswegs *jedem* separablen Raum eine Ordinaldimension zuordnen muss (es wird sich sogleich ergeben, dass beispielweise der Hilbertsche Raum H keine Ordinaldimension besitzt).

Die Klasse derjenigen separablen Räume, denen eine Ordinaldimension zukommt, bildet, wie man leicht einsieht einen (in Bezug auf das System aller separablen Räume) kompletten Bereich. Man zeigt ferner durch "transfinite Induktion" dass diese Klasse in jedem kompletten Bereich als Teilbereich enthalten ist, also den kleinsten kompletten Bereich bildet.

Mit Rücksicht auf die oben formulierten Ergebnisse schliessen wir daraus: *Einem separablen Raum kommt eine Ordinaldimension dann und nur dann zu, wenn derselbe mit einer Teilmenge eines abzählbar-dimensionalen kompakten Raumes homöomorph ist. Unter den kompakten Räumen stimmen also die mit einer Ordinal-dimension versehenen Räume mit den abzählbar-dimensionalen überein.*

Darin ist u.a. die Aussage enthalten, dass der Hilbertsche Raum nicht "dimensionierbar" ist, d.h. keine Ordinaldimension besitzt.

Die Ergebnisse der drei letzten Paragraphen zusammenfassend, können wir jetzt sagen:

Die kompakten abzählbar-dimensionalen Räume und die Teilmengen dieser Räume sind im Wesentlichen (d.h. von topologischen Transfor-

mationen abgesehen) *jene Mengen, deren dimensionelle Struktur mit Hilfe der Rekursionsprinzipien von* BROUWER, MENGER *und* URYSOHN *erfassbar ist.*

6. Zum Abschluss sei auf ein Theorem hingewiesen, das eine neue Charakterisierung der abzählbar-dimensionalen Mengen liefert, welche ganz allgemein für beliebige separable Mengen gilt.

Wir knüpfen an den Satz aus der Theorie der endlich-dimensionalen Mengen an [10]), welcher besagt, dass eine separable Menge (für ein endliches n) dann und nur dann höchstens n-dimensional ist, wenn sie sich als eindeutiges beiderseits stetiges Bild [11]) einer (linearen) nulldimensionalen Menge N darstellen lässt in der Weise, dass jeder Punkt von R höchstens $n+1$ Punkten von N als Bildpunkt zugeordnet wird.

Für abzählbar-dimensionale Mengen gilt das folgende Analogon dieses Satzes: *Unter den separablen Räumen sind die abzählbar-dimensionalen durch die Eigenschaft charakterisiert, dass sie sich aus nulldimensionalen Mengen durch eindeutige beiderseits stetige Abbildungen erzeugen lassen, bei welchen jedem Punkt des Bildraumes nur endlich viele Urbilder entsprechen.*

Ist insbesondere R ein kompakter abzählbar-dimensionaler Raum, so kann er durch eine Abbildung von der angeführten Eigenschaft aus der Cantorschen nirgends-dichten lineairen Menge C gewonnen werden. Indem wir die Terminologie von ALEXANDROFF und VIETORIS benützen, können wir auch sagen:

Alle kompakten abzählbar-dimensionalen Räume werden durch oberhalbstetige Zerlegungen der Cantorschen Menge C in *endliche Teilmengen* geliefert. Umgekehrt erzeugt jede derartige Zerlegung einen abzählbar-dimensionalen Raum.

[10]) Vgl. HUREWICZ, diese Proceedings, **29**, (1926), S. 1014—1017.

[11]) Wegen der Definition der beiderseitigen Stetigkeit bei Abbildungen, welche nicht ein eindeutig sind, vgl. loc. cit. [10]).

Chemistry. — *Osmosis of ternary liquids, General considerations.* VII.
By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of November 24, 1928).

Congruent and incongruent osmosis; the membrane.

In the preceding communication (Gen. VI) we have seen that there is a great difference in the behaviour of a membrane, transmitting only one substance and a membrane, transmitting more substances. We saw among other things:

a substance S diffuses through a membrane $M(S)$ in congruent direction viz. from smaller towards larger $O.S.A.$;

a substance S can diffuse through a membrane $M(n)$ in both directions namely congruently or incongruently; it depends on the nature of the membrane which of these directions will occur.

We might express this in the following way:

a membrane, which transmits only one single substance, is "passive", a membrane, which transmits more substances, is "active" with respect to the direction in which a substance diffuses.

We shall refer to this later on.

The three isotonic curves, which in fig. 2 (Gen. VI) run through point 1 divide the triangle into six fields; we call these field p, field q, etc. in accordance with the points p, q etc. situated in these fields. The six osmotic systems:

$$\left.\begin{array}{lll} L_1 \mid L_p & L_1 \mid L_q & L_1 \mid L_r \\ L_1 \mid L_s & L_1 \mid L_t & L_1 \mid L_u \end{array}\right\} \quad \ldots \ldots \quad (1)$$

which we have discussed already in Gen. VI, we shall call system p, system q etc. The $D.T.$'s (diffusion-types) of each of these systems are found in scheme I (Gen. VI); in this scheme namely we have indicated for each separate system, which $D.T.$ is incongruent and, therefore, not possible. Consequently N°. 5 disappears in system p, N°. 6 in system q, N°. 2 in system r etc.

In the osmotic system

$$L_1 \mid L_p \quad \ldots \ldots \ldots \ldots \quad (2)$$

all $D.T.$'s, are possible except the incongruent one, viz. N°. 5. It appears, however, from the deduction that we have to take this "possible" only in the sense: from a thermodynamical point of view there is no objection

to the occurrence of these seven $D. T.$'s. For this reason we shall call them: "admissible" $D. T.$'s.

Of course an infinite number of systems p must exist; we namely may imagine the right-side liquid p of (2) in any point of field p and besides in each of these systems different membranes. All these systems have the same incongruent $D. T.$ N⁰. 5.

We now imagine in (2) a definite membrane and a definite liquid p; then also a definite $D. T.$ will occur, e.g. N⁰. 4. When liquid p now travels through field p, then it may be that all these systems yet have the same $D. T.$ N⁰. 4. Then only N⁰. 4 occurs of the seven "admissible" $D. T.$'s.

We can also imagine, however, that in one part of field p the $D. T.$ N⁰. 4 obtains and in an other part e.g. N⁰. 3. Then only two of the seven "admissible" $D. T.$'s will occur.

If in (2) we now replace the membrane by another, we may imagine that e.g. the $D. T.$ N⁰. 1, 2 and 4 will obtain.

From this it becomes clear that we can say:

The composition of the liquids determines which $D. T.$ will be incongruent; which of the "admissible" $D. T.$'s will obtain, is not only determined by the composition of the liquids but also by the nature of the membrane.

These considerations now lead us to the question:

is it possible to find some further connection between the respective influences of the membrane and the composition of the liquids on the $D. T.$

Before we can enter upon a discussion of this question it will be first necessary to consider two more special cases of the osmosis.

A. We take the osmotic system:

$$\left.\begin{array}{ccc} & L \mid L' & \\ \longleftarrow \alpha X & \longleftarrow \beta Y & \longleftarrow \gamma W \end{array}\right\} \quad . \quad . \quad . \quad (3)$$

in which α mol. X, β mol. Y and γ mol. W diffuse in the direction of the arrows. If we take α, β and γ infinitely small, they will have to satisfy (comp. Gen. VI):

$$\alpha K_x + \beta K_y + \gamma K_w > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We now imagine the special case that both liquids of (3) have the same $O. W. A.$, then no water can possibly diffuse through a membrane $M(W)$. Is this also the case, however, when we take a membrane $M(XYW)$?

When the liquids have the same $O. W. A.$ then $K_w = 0$; in (4) the last term will disappear and we get:

$$\alpha K_x + \beta K_y > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that γ now needs not satisfy a single condition; it is only necessary

that α and β shall satisfy (5). Consequently (4) or (5) can be satisfied not only by $\gamma = 0$ but also by pos. and neg. values of γ.

So it is possible that a membrane $M\,(X\,Y\,W)$ exists, through which no W will diffuse; in general, however, water will diffuse and the nature of the membrane will determine in which direction it will go. Consequently we may say:

When two liquids have the same $O.W.A.$, then W does not diffuse through a membrane $M\,(W)$ but as a rule it does diffuse through a membrane $M\,(n)$.

As, however, (5) must be satisfied, both terms cannot be negative at the same time. This means (Comp. Gen. VI) that X and Y do not diffuse incongruently at the same time. So we find:

when two liquids have the same $O.W.A.$, then $n-1$ other substances cannot diffuse incongruently at the same time.

It is clear that the preceding is not only valid for the substance W, but for any substance S.

B. When the $O.X.A.$, $O.Y.A.$ and $O.W.A$ differ on both sides of the membrane, then there are seven "admissible" $D.T.$'s; there are transitiontypes between them, however, which occur when one of the substances does not happen to pass through the membrane, If e.g. N⁰. 1 and N⁰. 3 of scheme I are both "admissible" $D.T.$'s, then W diffuses

SCHEMA I

	X	Y	W
1.	$\longleftarrow$	$\longrightarrow$	$\longleftarrow$
2.	$\longleftarrow$	$\longrightarrow$	——
3.	$\longleftarrow$	$\longrightarrow$	$\longrightarrow$

in 1 towards the left and in 3 towards the right. Now we may imagine as a transition-state that the water diffuses neither towards the left nor to the right; we then get the transition $D.T.$ N⁰. 2, in which the arrow sub W has been replaced by a dash.

If in (4) we put $\gamma = 0$, so that this passes into (5), we see that the transition-$D.T.$ is "admissible". Consequently we may say:

When two substances have a different $O.W.A.$, then W must always diffuse through a membrane $M\,(W)$, but not always through a membrane $M\,(n)$.

If accidentally W does not diffuse through a membrane $M\,(n)$, then the $n-1$ other substances cannot diffuse incongruently at the same time.

Of course the above also obtains for every substance S.

C. It appears among other things from the following remarks that the phenomena, discussed above, can really take place.

Experimentally (comp. Gen. VI) we have found that in a definite system the diffusion went through three different membranes according to

$D. T.$ N^0. 1 of scheme I, but through three other membranes according to N^0. 3. We can now imagine that there may be membranes, for which N^0. 2 is valid.

We found the same thing for some other systems too (comp. Gen. VI).

We may find the transition-type also by changing not the membrane, but the composition of one or of both liquids.

We take e.g. the two systems:

$$L \mid L_a \qquad L \mid L_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which L_a and L_b have different compositions; they have the same L, however, and the same membrane. We now found (comp, Exp. I—III) that in the one system the diffusion took place according to N^0. 1 and in the other system according to N^0. 3 of scheme I. Consequently between L_a and L_b a liquid must be situated, for which the $D. T.$ N^0. 2 (scheme I) obtains.

If we take an arbitrary osmotic system and leave it to itself, every liquid will travel along its path. Now it has been proved experimentally (Exp. I—III) that the $D. T.$ may change during the osmosis. On one part of the path e.g. N^0. 1 is valid and on the other part N^0. 3 of scheme I. So there is also a moment in which N^0. 2 is valid and consequently no water diffuses.

Similar systems, viz. those in which one of the substances does not pass through the membrane, generally exist only for a single moment. Because of the diffusion of the other substances namely, the liquids change their compositions and this change is generally of a nature, that all substances will diffuse again at once.

D. It appears from the above that the diffusion of water (or another substance) will be different, when different membranes viz. $M(W)$ or $M(n)$ are used. We shall compare both cases with one another.

1. The membrane $M(W)$. When the membrane only transmits W, then it only depends on the $O. W. A.$ of the two liquids whether the water will diffuse or not and in which direction this will take place. The $O. W. A.$ of a liquid only depends on its composition; we have [comp. (6) Gen. VI]

$$O. W. A. = \xi_w = -\zeta + x\frac{\partial \zeta}{\partial x} + y\frac{\partial \zeta}{\partial y}.$$

With the aid of this we found;

I. when two liquids have the same $O. W. A.$, no W diffuses;

II. when two liquids have a different $O. W. A.$, W does diffuse;

III. W diffuses congruently viz. from smaller towards greater $O. W. A.$

2. The membrane $M(n)$. It appears from our considerations of systems, in which more substances pass through the membrane, that the rules I—III mentioned above do not obtain here. In A we have already

seen namely that W can indeed diffuse through a membrane $M(n)$, notwithstanding the $O.W.A.$ of both liquids is the same; consequently rule I is no longer valid.

In B we saw that sometimes no W diffuses through a membrane $M(W)$, notwithstanding the $O.W.A.$ of the two liquids is different, and in Gen. VI we have seen that a substance W can diffuse through a membrane $M(n)$ as well congruently as incongruently. Consequently the rules II and III are not valid any longer either.

So the $O.W.A.$ plays a direct part with a membrane, only transmitting W, but no more when at the same time other substances may diffuse also. Then the diffusion of the water not only depends on the $O.W.A.$ but also on the other diffusing substances and besides on the nature of the membrane.

We may assume, however, that a function exists, which plays a part with respect to the diffusion of W through a membrane $M(n)$ viz. the function ξ_w corresponding to that of the $O.W.A.$ with respect to the diffusion of W through a membrane $M(W)$.

We shall call this the $O.W.n.A.$ of a liquid; here the n indicates that n substances diffuse. This function, however, not only depends on the composition of the liquids, but, as it is valid for a membrane $M(n)$, also on the nature of the membrane. We shall refer to this when discussing the adsorption etc. of membranes.

We now choose this $O.W.n.A.$ in such a way that the diffusion of water through a membrane $M(n)$ we comply with:

I^n. when two liquids have the same $O.W.n.A.$, no W diffuses;

II^n. when two liquids have different $O.W.n.A.$, W does diffuse.

III^n. W diffuses from smaller towards greater $O.W.n.A.$

In connection with our previous considerations it is clear now:

a. when two liquids have the same $O.W.A.$, they generally will have a different $O.W.n.A.$:

b. when two liquids have a different $O.W.A.$, they may yet happen to have the same $O.W.n.A.$;

c. when a liquid p has a greater $O.W.A.$ than a liquid q, p may yet have a smaller $O.W.n.A.$ than q.

In A we have already seen namely that water can pass through a membrane $M(n)$ notwithstanding the liquids have the same $O.W.A.$; then they have a different $O.W.n.A.$ (comp. *a*.).

We saw in B that sometimes no water passes through a membrane $M(n)$ notwithstanding the liquids have a different $O.W.A.$; then they have the same $O.W.n.A.$ (comp. *b*.).

In Gen. VI we saw that water can diffuse through a membrane $M(n)$ from a liquid p towards q, notwithstanding p has a greater $O.W.A.$ than q; then the liquid p has a smaller $O.W.n.A$ than q (comp. *c*.).

When two liquids have the same $O.W.A.$ and consequently no water

passes through a membrane $M(W)$, we call these "isotonic" with respect to W.

When two liquids have the same $O.W.n.A.$ and consequently no W passes through that same membrane $M(n)$, we shall call these "iso-n-tonic" or "isentonic" with respect to W.

It appears from the above:

when two liquids are isotonic with respect to W, they are generally not isentonic;

when two liquids are not isotonic with respect to W, then they may yet happen to be isentonic.

In Gen. VI we have deduced:

all liquids having the same $O.W.A.$ as a liquid g are situated on a curve $a\,b$, running through point g (comp. fig. 1 Gen. VI); we have called this an "isotonic W-curve".

We now may say also:

all liquids, which have the same $O.W.n.A.$ as a liquid g are situated on a curve going through point g; we call this the "isentonic W-curve".

It is clear and we shall refer to this later on that the isotonic and isentonic W-curves, running through the same point will generally vary a good deal.

All that has been said above for the diffusion of water obtains of course also for the diffusion of other substances; with the diffusion of an arbitrary substance S we then can speak of the $O.S.n.A.$ and the isentonic S-curve.

We shall see besides how these isentonic curves may serve us in order to get a better insight in the diffusion through a membrane $M(n)$.

E. We may deduce some results of this and preceding communications also in an other way. For this purpose we consider a system

$$L \mid L' \qquad (7)$$

in which two or more membranes have been placed between the liquids; we imagine them to lie at some distance from one another so that they can funtion independently from one another.

If in (7) we imagine two different membranes $M(S)$ then we may put the question:

Can the substance S diffuse through the one membrane towards the left and through the other towards the right?

If we assume this to be the case, then a current of S would arise in the system, running through the one membrane towards the left and

through the other membrane towards the right; we call this a "circular current".

If the quantity of S, diffusing through these membranes, is different however, then both liquids change their compositions and at last the system comes in a state of equilibrium; the circular current of S has then stopped.

The surfaces of the membranes however may be treated in such a way that through the one as much S runs towards the left as through the other towards the right; then the liquids do not change their compositions and we get an eternal circular current of S. As we take, however, that this is not possible, we may conclude, therefore:

a substance S diffuses through all membranes $M(S)$ in the same direction.

Previously we have called this the congruent direction of S.

If in (7) we imagine a membrane $M(n)$ which transmits the substances X, Y, Z etc. we may put the question:

Can all these substances run through the membrane incongruently at the same time?

When the substance X diffuses incongruently, consequently in a direction opposite to the one through a membrane $M(X)$, we can regulate the surface of this membrane in such a way that as much X runs through $M(X)$ as through $M(n)$.

We are able to do the same for every incongruently diffusing substance, viz. for Y with a membrane $M(Y)$, for Z with a membrane $M(Z)$; etc.

When all substances diffuse incongruently, we have, therefore, a system in which at every moment as much X, Y etc. run through the membranes $M(X)$, $M(Y)$ etc., in congruent direction as through the membrane $M(n)$ in incongruent direction. Then the composition of the liquids would not change and we should have n eternal circular currents. So it follows from this:

all substances cannot diffuse incongruently at the same time; consequently the incongruent $D.T.$ is not admissible. We can also put the question:

Can the substances diffuse according to the congruent or to one of the mixed $D.T$'s?

For every incongruently diffusing substance we may place a membrane transmitting the same quantity of this substance in congruent direction. As, however, we are not able to put an end to the diffusion of the congruently diffusing substances, the liquids consequently change their compositions, so we cannot get eternal circular currents now. From this point of view it follows in the same way as before:

the congruent and mixed $D.T.$'s are admissible.

When the diffusion through a membrane $M(n)$ takes place according

to a mixed $D.T.$, we can alter this $D.T.$ with the aid of an other membrane.

If one of the incongruently diffusing substances is e.g. water, we can do this with a membrane $M(W)$ with such a surface that more W runs in congruent direction than in incongruent direction.

If we regulate the surface in such a way that the same quantity of W runs through both membranes, this combination of membranes functions, therefore, as a single membrane, through which no W diffuses in that moment. At that moment then the liquids are not "isotonic" but "isentonic" with respect to water; they then have not the same $O.W.A.$ indeed but the same $O.W.n.A.$

(To be continued).

Leiden, Lab. of Inorganic Chemistry.

Geology.— *On Tertiary Rocks and Foraminifera from North-Western Peru.* By L. RUTTEN.

(Communicated at the meeting of September 29, 1928.)

The Institute of Geology of the University of Utrecht was presented by the Bataafsche Petroleum Maatschappij with a collection of rocks from N. W. Peru made by its geologists in the years 1924 and 1925. These rocks are worthy of notice in that they afford new points of view with reference to the Peruvian Tertiary and contain remarkable fossils.

The literature of the Tertiary of N. W. Peru is not extensive. Geological data are to be found in publications by J. GRZYBOWSKI [1]), V. F. MARSTERS [2]), R. A. DEUSTUA [3]), BEEBY THOMPSON [4]) and J. J. BRAVO [5]). Of much greater importance is the minute description of the region by T. O. BOSWORTH [6]). Palaeontological treatises upon the Tertiary of N.-W. Peru are of much earlier date. To the 19th century belong publications by A. D'ORBIGNY [7]), W. M. GABB [8]), E. T. NELSON [9]), and J. GRZYBOWSKI (l.c.). The fossils collected by BOSWORTH have been examined by H. WOODS, H. L. HAWKINS, T. WAYLAND VAUGHAN and J. CUSHMAN [10]), short memoirs on the eocene fauna of N.-W. Peru have

1) J. GRZYBOWSKI, Die Tertiärabl. d. nördl. Peru etc. N. J. f. Min. B. B. 12. 1899. p. 610—662.

2) V. F. MARSTERS, Informe preliminar sobre la zone petrolifera del Norte del Peru. Bol. Cuerpo Ingenieros de Minas y Aguas. Lima. N⁰. 50. 1907.

V. F. MARSTERS, The Physiography of the Peruvian Andes. Ann. New York Acad. Sc. 22. 1912. p. 225—228.

3) R. A. DEUSTUA, Estado actual y porvenir de la industria petrolifera en el Peru. Lima. 1912.

R. A. DEUSTUA, El petroleo en el Peru. Lima. 1921.

4) BEEBY THOMPSON, The geology of Northern Peru. Geological Magazine. 1913. p. 233—234.

5) J. J. BRÁVO, Reconocimiento de la región costanera de los departementos Tumbes y Piura. Archiv. Asoc. Peruana para el progreso de las ciencias. I. 1921. p. 15—19.

6) T. O. BOSWORTH, Geology of . . . . the Northwest Part of Peru. London, 1922.

7) A. D'ORBIGNY, Paléontologie du Voyage dans l'Amérique méridionale. Paris. 1842.

8) W. M. GABB, Descr. of new species of South American fossils. Amer. J. of Conchology 5. 1869. p. 263.

W. M. GABB, Description of a collection of fossils, made by Dr. A. RAIMONDI in Peru. J. Acad. Nat. Sc. Philadelphia. (II). 8. 1877. p. 264.

9) E. T. NELSON, On the molluscan fauna of the later Tertiary of Peru. Transact. Connecticut Acad. of Sc. a. Arts II. 1873. p. 186—206.

10) H. WOODS, Eocene and Miocene Mollusca; H. WOODS, Eocene Crustacea; H. L. HAWKINS, An echinoid from the Eocene; T. WAYLAND VAUGHAN, Eocene, Corals; J. CUSHMAN, Eocene foraminifera; in: T. O. BOSWORTH, l.c. p. 51—142.

lately been written by H. DOUVILLÉ [1]), and E. W. BERRY [2]). Tertiary foraminifera (to be discussed lower down) have been made known by J. CUSHMAN (l.c.), C. LISSON [3]), A. TOBLER [4]) and H. DOUVILLÉ [5]), while foraminifera from the adjacent part of Ecuador have been described by T. WAYLAND VAUGHAN [6]). Finally E. W. BERRY [7]) has examined miocene plants.

Several results of the older researches deserve mention here. The tertiary deposits of N. W. Peru are of an enormous thickness. The older inquirers were not quite aware of this: GRZYBOWSKY reports a minimum thickness of 700 m; THOMPSON and MARSTERS mention a minimum thickness of about 1500 m. BOSWORTH, whose observations are no doubt the most numerous and the most detailed of all, reports a thickness of from 4500 to 7500 m:

Zorritos	formation more than 1500 m	(Miocene ?).
Lobitos	formation more than 1500 m	Eocene.
Negritos	formation more than 2100 m	Eocene.

Neither did geologists agree in the course of time as to the age of the Tertiary. According to GABB several formations occur near Paita, among which there is certainly the Pliocene. NELSON speaks of "Later Tertiary" near Zorritos. According to GRZYBOWSKI the following subdivisions may be distinguished:

Payta Stufe	Pliocene.
Talara Stufe	Miocene.
Zorritos Stufe	Lower Miocene.
Heath Stufe	Lower Miocene.
Ovibio Stufe	Upper Obliocene.

The results of BOSWORTH's inquiry have already been given higher up. For his age-estimations he relies upon the palaeontological determinations

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1) H. DOUVILLÉ, l'Eocène au Pérou. C. R. Acad. Sc. Paris. **171**. 1920. p. 1345—1347. C.R. somm. Soc. Géol. France. **14**. 1921. p. 193.

2) E. W. BERRY, A new Hercoglossa from the Eocene of Peru. Amer. J. Sc. **6**. 1923. p. 427—431.

3) C. LISSON, Contr. al estudio de algunos foraminiferos terciarios provenientes de la región del Norte del Peru. Arch. Asoc. para el progreso de las Ciencias. Lima I. 1921. p. 52—55.

4) A. TOBLER, Neue Funde von Grossforaminiferen in der Nordperuanischen Küstenregion. Eclogae geologicae Helvetiae. **20**. 1927. p. 415—422.

5) H. DOUVILLÉ, Revision des Lepidocyclines. Mém. Soc. Géol. de France. Nouv. Série. I. **2**. 1924.

6) T. WAYLAND VAUGHAN, Foraminifera from the Upper Eocene deposits of the coast of Ecuador. Proc. Nat. Acad. Sc. Washington. **12**. 1926. p. 533—535.

7) E. W. BERRY, Miocene fossil plants from North Peru. Proc. U.S. Nat. Museum. **55**. 1919. p. 279—294.

of WOODS and others. That the results arrived at by GABB, GRZYBOWSKI and BOSWORTH differ, appears from the fact that according to BOSWORTH only Eocene occurs in the territory of Paita and Talara, while GABB and GRZYBOWSKI report Pliocene. On the other hand BOSWORTH indicates between Punta Sal and Paita (see map on p. 934) only "Zorritos formation", whereas TOBLER also describes eocene foraminifera from this region and points out that BOSWORTH'S map is certainly too schematic for this region. On the one hand it, therefore, appears from the older inquiries that the network of observations is certainly not close enough, and on the other hand that it is often difficult to fix the correct age of tertiary deposits.

That the Negritos- and Lobitos-formations must be referred to the eocene, seems to be generally accepted at present, which is to no little degree owing to the occurrence of Venericardia planicosta Lmck. Now it imports us to know that, according to WOODS, the eocene strata of N. W. Peru show palaeontological affinity on the one side with those of California (Medanos and Tejon series) and on the other side to those of the Gulf-region (Clayborne and Wilcox series).

We do not know whether in Peru continuous deposition has taken place from Eocene to Miocene or Pliocene. THOMPSON speaks of an unconformity prior to the Miocene ; BOSWORTH also considers this possible. It is a positive fact, however, that after the Miocene and before the Quaternary a stupendous disturbance has taken place of the enormously thick tertiary deposits. BOSWORTH holds that these disturbances were rather faulting than folding processes. The whole region of N. W. Peru should, according to this author, be considered as a gigantic "crush-breccia". The orogenetic movements have been succeeded by stupendous denudation, followed by quaternary transgression. The quaternary deposits have at various times been lifted up vertically, so that nowadays different littoral terraces are superposed upon each other.

The Tertiary of N. W. Peru consists mainly of shales, sandstones, and littoral gravel ; all of them may contain a certain amount of lime. The boulders of the littoral gravel attain a diameter up to 15 cm. Frequently a coalbearing substance occurs in the shales (especially in the Negritos-formation). The occurrence of true lignite-layers I find recorded only for the upper part of the Zorritos-formation (BOSWORTH, BERRY 1919).

Taking all in all the Tertiary of N. W. Peru displays in many respects affinity with that of California. This affinity is of a palaeontological character (DOUVILLÉ 1920, WOODS), of a petrographic character (DOUVILLÉ 1920, oil occurs in either), of a stratigraphical nature in that in either region the Tertiary is very thick, and of a tectonic character in that in either region the Tertiary has been considerably disturbed.

It will be well to discuss the rocks of the "Bataafsche Petroleum Maatschappij" in geographical order.

The rocks from the northern part of the region, between Boca Pan and Casitas (locality 28, 29, 36) are of slight importance. A number of samples of clay-shale of the first locality originate from the "Heath-formation". Inter se they agree well. These rocks easily turn to silt, their silt-rests contain a few small foraminifera, stratigraphically insignificant. We also

find traces of fish-rests and scarce grains of quartz and plagioclase. The absence of diatoms is remarkable. From this locality was also derived a peculiar, greenish-black rock, composed of granules of glauconite, bound together by a cementing material of siderite (D. 10477)[1]. I had the disposal of two samples of clay-shale from locality 29. They were held

[1]) The D-numbers refer to the collection of slides of the Geol. Institute of Utrecht.

to be eocene; they contain in their scanty silt-rests also traces of insignificant foraminifera and grains of quartz and plagioclase. In one of the samples there are scarce, centric Diatoms. The rock of locality 36 is a quartz-sandstone, built up of grains of "old" quartz with very scarce rutile, zircon, and turmaline and with siliceous cement.

The rocks described are totally unfit for the determination of age. If the glauconite-siderite rock is spread fairly horizontally, it will form an excellent guide-layer for the terrain.

A rather large number of samples were collected in the neighbourhood of Punta Sal. Eocene limestones with Isolepidina R. Douvillei and Orthophragmina found in these localities have been described by TOBLER (l.c.). In different places stratigraphically neutral rocks have been collected here: a gasteropodlime (43), a gray and a brown argillaceous shale with traces of small foraminifera and without diatoms (45) and a calcareous marl concretion, intensely crystalline without clastic material (61). However, from locality 46 a gray strongly recrystallized limestone (D. 10473) is derived, which contains besides numerous Lithothamnia traces of Orthophragmina and is certainly of eocene age. This is the only Orthophragmina-rock that was found in the collection. Furthermore some five samples (45) of a brownish gray limesandstone (D. 10488—10492) were collected, all of which contained Operculina nummulitiformis n.s. The majority of the clastic grains in these rocks are "old" quartz. There are further a limestone (60), consisting almost entirely of shales of Operculina nummulitiformis (D. 10495) and four brown to brownish-red limes (44), all abounding with this fossil (D. 10480—10487, 10493, 10494). In one of these rocks (D. 10483) there occur also very irregular, scarce, small Lepidocyclines apparently stunted in their growth. All these rocks must be of eocene age. From locality 42 originates a fragment of coral-lime, besides two fragments of a sandy, brownish-yellow Lepidocyclina-limestone. The clastic material consists again mainly of dusty quartz, some plagioclase, muscovite and pieces of a quartzitic rock (D. 10591, 10597—10599). The Lepidocyclinae are closely allied to L. R. Douvillei Lisson, but differ from it in possessing numerous columns; they must be referred to a new variety (var. armata). Of locality 44 there is still a calcareous-sandstone, containing Lepidocyclines and a few individuals of Operculina nummulitiformis.

We see, then, that eocene rocks abound in the vicinity of Punta Sal.

Farther into the interior (53) a sandy-marly shale has been collected. It contains many quartz-splinters and traces of small foraminifera. In the terrain it was held to be eocene.

From Quebrada Mancora (55) two rocks have been derived: one eocene(?), a lime-free, fossil-free clay-shale, and a polygene, fossil-free calcareous-sandstone containing much "old" quartz and plagioclase.

LISSON and TOBLER (l.c.) have already recorded rocks from Los Organos (69) with Lepidocyclina R. Douvillei. In the collection of the "Bataafsche Petroleum Maatschappij" there are a calcareous-sandstone with an

uninforming Cristellaria (D. 10478) and three calcareous-sandstones (D. 10600—10602), which contain numerous specimens of Lepidocyclina R. Douvillei var. armata. The clastic material here also is for the greater part dusty quartz, rather many grains of quartzitic rocks and possibly radiolarite.

In the vicinity of Vichayal (78) seven samples of "Pariñas sandstone" have been found (D. 10447, 10496—10507, 10592—10596). They are slightly calcareous sandstones prone to crumble, and coloured yellowish-brown by weathering. The clastic material they contain is composed of old quartz, plagioclase, orthoclase, much green amphibole, a little zircon, ore and traces of garnet. In the silt-residue we found: traces of scales of fish and some lamellibranchs, a few specimens of Operculina nummulitiformis and a good many specimens of a small Lepidocyclina related to Lep. R. Douvillei. The fossils of Vichayal, however, exhibit some constant deviations from this species; they will be described as Lepidocyclina Vichayalensis n.s.

Whereas the rocks from Vichayal fairly agree with those from the vicinity of Punta Sal and Los Organos, higher up in the Rio Chira some samples of quite a different type have been collected. From Miraflores (83) a "white bed on top of iris shales" is derived (D. 10470). It appears to be a dacite-tuff without organisms with an isotropic matrix, in which there are numerous splinters of lucid quartz, plagioclase, and sparse biotite. It is genetically related to a white sandstone from Sullana (86), whose components are besides very lucid, zonary plagioclase, and bipyramidal quartz with glass-xenoliths, also a few pieces of "old" quartz and quartz-schist. Furthermore the sandstone contains many loosened fragments of effusive matrix. Finally a rock from the Chira-valley is non-typical: (84 Station Nomara). It is a brown shale ("Heath-shale") in whose scarce washing-residues some small foraminifera are recognized.

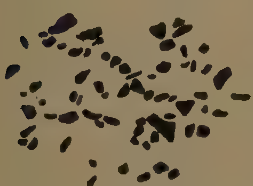
Another quite new type of rock has been collected between Chira-, and Piura-river. It easily turns to silt, is very light and is clayey to the touch. In its minimal ooze-residues only quartz can be identified. In a preparation made of the scrapings of the rock numberless marine, centric diatoms occur. The rock is a yellow diatomite.

A very large number of rocks, chiefly of no importance, have been collected in the close vicinity of Paita. Sixteen samples of "Heath-shales" from locality 108 are fairly alike inter se. They readily turn to silt, are brownish-gray to yellowish-gray and finely argillaceous to sandy; often with many small particles of coal; gypsum occurs in narrow veins and on fault-planes. Of minerals we observe in the washing-residues especially splinters and dust of old quartz and plagioclase. The marly shales always have a small content of stratigraphically insignificant foraminifera: Textularidae, Nodosaridae, Rotalidae, and Globigerinidae. Rarely do we find rests of Lamellibranchs and Gastropods; there are nearly always small scales and toothlets of fish. Diatoms are ever absent. The rocks collected in the localities 109 and 110 are completely alike.

Various other rocks from this region adjoin the "Heath-shales". A yellow, calcareous-marl from 110 (D. 10446) with small Rotalidae and Textularidae is only a somewhat more calcareous variety of the marls. In the shales concretions occur collected in all localities mentioned (108, 109, 110) (D. 10443, 10444, 10445, 10459, 10468). They are all yellow, dolomitic, porous limestones, almost without clastic material, and consisting of small loosely adhering rhomboheders of calcite and dolomite. Occasionally they contain small foraminifera. Lastly to the group of the preceding rocks belongs a yellow lime from 110, with numberless small Textularidae (D. 10469).

The rocks of the "Heath-formation" that have been described, are not fit for the determination of age, but the shales and their concretions are easily discernible rocks. In the field they were considered to be oligocene. Only one rock in the vicinity of Paita was considered eocene ; a calcareous-sandstone of the "Disco-formation" (locality 110). It contains Operculina nummulitiformis and Lepidocyclinae ; the clastic material it contains is old quartz, muscovite and ?porphyrite : so it bears a great resemblance to the eocene rocks found more towards the north.

Another group of shales with concretions found more towards the south, have been referred to the eocene by the geologists of the "Bataafsche Petroleum Maatschappij", and classed under the "Playa ancha series". The direct relation between "Heath" and "Playa ancha" could not be established. Microscopical examination has made out that the two formations can indeed be distinguished the one from the other, but cannot afford a positive confirmation as to the age of the "Playa ancha series". The shales (128, 129) are grayish-brown, porous, lime-free, very easily silting rocks ; in the scanty residues are found quartz, plagioclase, and a few radiolaria, but foraminifera are lacking. It may be that volcanic glass occurs in the residue. Diatoms are not scarce. The concretions from the Playa ancha shales (125, 128, 129) also differ from those from the "Heath": they contain namely many more mineral splinters and lack the rhomboheder-structure. Small, irrelevant foraminifera and calcified radiolaria may occur in them. It is very striking that in the Playa ancha concretions the mineral fragments (old quartz, plagioclase and muscovite) are very sharp splinters without a trace of rounding (D. 10461—10466). In this they forcibly remind us of the "splinter-sand" from the recent-desert of N.W. Peru, described by BOSWORTH, for comparison we refer to Fig. 83 of BOSWORTH and the annexed figure. However the mineral splinters in the Playa ancha-concretions are very much smaller than those from the recent splinter-sand. It seems to me, that the sharp-edged mineral-splinters in the Playa-ancha concretions imply, *that a desert-climate prevailed already during their deposition in N.W. Peru.*



The difference between "Heath"-, and "Palya-ancha"-rocks may be briefly summarized here.

Heath : marly shales with not infrequent foraminifera, without radiolaria or diatoms ;
dolomitic calcareous concretions with rhomboheder-structure, almost without clastic material.

Playa-ancha : calcium-free, porous shales, well-nigh without foraminifera, with radiolaria and diatoms ;
calcareous concretions without rhomboheder-structure with much micro-splintersand.

Locality 106 still furnished two samples of "Playa-ancha shales", which however are different rocks. The one is a yellowish-white diatomite with many centric diatoms and a few radiolaria, the second is a yellowish-white, crumbling marl with many isolated carbonate-rhomboheders, with pieces of quartz, plagioclase and garnet and with traces of diatoms.

Of locality 118 (Tortugas-Pta Perico) a crumbling sandstone with "old" minerals is present ; of 124 (close to Silla de Paita) an irrelevant, yellowish-brown calcareous-sandstone.

The last district where rocks have been collected, is the surroundings of the Cerro Illescas. From locality 142 (Yapate) three very typical rocks are derived : the first two white, very soft rocks, which appear to consist almost entirely of intact specimens and fragments of centric diatoms, the third is a much more compact rock, whose scrapings contain besides diatoms also isotropic or faintly aggregate-polarizing flakes and grains. The former rocks are typical samples of diatomite, the latter is a siliceous diatom-bearing tuff (D. 10448, 10449). Completely comparable rocks have been taken from locality 149 : white, gray, and green varietal forms of marine diatomite ; in one sample occasional Textularidae and Cristallaria also occur. Also slightly differing rocks have been collected at the last-named locality : a yellowish-white, crumbling limestone, still containing many diatoms, and a greenish white, siliceous marl (10450), with very scarce diatoms. A white diatomite also originates from 150. In the environment of the named localities the following rocks have been collected. Near 147 (D. 10457) a lime sandstone with splinters of quartz (?young-effusive) as clear as water, plagioclase, biotite and muscovite ; near 151 a porous, sandy limestone (D. 10460) with rounded granules of old quartz, biotite, muscovite and plagioclase and with undeterminable remains of organisms ; near 152 a very peculiar white, oölithic-like rock. It appears to consist of white globules of silicified diatomite, kept together by siliceous cement (D. 10459). It is on this account a "silicified sand of diatomite".

A good many rocks, differing from the preceding ones, have been found near the coast. An opalized, chalcedonized sandstone (D. 10452) with hyaline quartsz-splinters, was derived from 137 (Cerro El Viejo). There are several rocks from 138; two peculiar calcified andesite-tuffs, and a mari with a marl-lime. The first-named (D. 10455, 10456) are white rocks with much fresh plagioclase of zonary structure and with splinters and hooks

of volcanic glass and pumice-stone: all kept together by a limy cement, which is beautifully crystalline and is orientated over large distances. Part of the marl easily turns to silt: it contains besides effusiva, fresh plagioclase, numerous small uninforming foraminifera and a rare small Lepidocyclina (D. 10453). The yellowish-white limestone (D. 10454) is rich in stratigraphically unimportant foraminifera. Locality 131 and 132 (D. 10479) furnished sandy marls, which contain clastic material, as glass-bearing, fresh plagioclase and volcanic glass, besides old quartz and muscovite; of remains of organisms we find rests of fish, small foraminifera, radiolaria and diatoms. In 133 and 136 isolated specimens of a new Lepidocycline have been collected, which will be described as Polylepidina variabilis n.s. From 134 two limestones have originated (D. 10522). The only clastic material contained in one of them is quartz, in the other also rolled-down pieces of greywacke. They contain numerous, small Lepidocyclines probably belonging to Isolepidina R. Douvillei var. armata.

Finally three limestones originate from locality 130 (D. 10604, 10605). They are quartz-bearing limestones with many specimens of Operculina nummulitiformis; one sample also contains small undeterminable Lepidocyclines.

The following remarks are suggested by the rocks described.

In part they open up a new vista into the Tertiary of N. W. Peru. This refers least to the composition of the "old" material in the tertiary formation. The minerals found (quartz, felspars, amphibole, mica, zircon, turmaline, ore and garnet) are such as are always concentrated in the decomposition of an "old" mountain-range; moreover their presence could be safely anticipated. The *presence of rolled-down radiolarite* alone would be striking (locality 69); but its presence could not be demonstrated with certainty.

The occurrence of much tuffogene material is new (dacitetuff in 83, dacitesandstone in 86, effusive quartz in 137, 147, andesite tuff in 138, marl with andesite material in 138, and limestones with andesite material in 131, 132). They suggest volcanic eruptions with dacitic-andesitic material at none too great a distance. The rocks, in which the effusive material occurs, are all considered to be Eocene; of some this view could be corroborated by palaeontological data. In the "Heath formation" (considered to be Oligocene), however, no effusive material could be detected. In this respect there is a contrast between N. W. Peru and S. Ecuador. SHEPPARD has just described a tuffogene material from the lower-oligocene "Ancon white sandstone" of Ecuador, while the Eocene is free from volcanic material there [1]). It should be observed, however, that SHEPPARD's publication does not give us decisive arguments for referring the white Ancon-sandstone to the oligocene.

[1]) G. SHEPPARD, The geology of Ancon point, Ecuador. J. of Geol. **36**. 1928. p. 113—138.

It is remarkable, that the shape of the clastic material in the concretions of the "Playa ancha shales" probably indicates the pre-existence of *a desert-climate in N. W. Peru during their deposition*. In this connection it should be observed that for part of the Californian Eocene (the Medanos, Series) BRUCE L. CLARK has also suspected on other ground the existence of a very dry hinterland [1]).

The most remarkable result no doubt is *the finding of depositions with marine diatoms on a large scale* [2]). Similar rocks were not known either from the Tertiary of N. W. Peru or from Ecuador or Chile. Sometimes they are siliceous diatomaceous-earths (localities 91, 106, 142, 149, 150, 152); sometimes they are shales, which contain marine diatoms (locality 29, the Playa-ancha shales); sometimes the diatoms occur in calcareous rocks (locality 131, 132). By the finding of these diatomaceous deposits the geological relationship between the Tertiary of N. W. Peru and California is once more substantiated. Whereas, however, in California the diatomaceous deposits are best developed in the Miocene (Monterey shales — although they are not absent in the Eocene (e.g. in the Coalinga Oil field) — the diatomaceous-deposits of N. W. Peru seem to be exclusively of eocene age. Anyhow, in the ?oligocene "Heath-formation" no further trace of diatoms was recognized.

The occurrence of marine diatomaceous-deposits in N. W. Peru is of twofold geological importance, viz. regionally and generally. Until recently North-American geologists were of the opinion that for an understanding of the genesis of the tertiary diatomaceous deposits of California — which have been formed in a temperate climate — it is required to assume that arctic currents supplied the diatoms from the northern part of the Pacific, where they are known to abound and that these diatoms succumbed in warm, more or less closed lagoons [3]). So a complete apparatus of hypotheses was laid under contribution to make the genesis of the Monterey-shales and facially similar deposits harmonize with the actually-geologic data. It is evident that the occurrence of similar Diatom-depositions at less than 10° from the aequator cannot be explained in this way. But at the same time also the data of "actual geology" have altered. Whereas CLARK could still write in 1921 that "diatomaceous oozes in any considerable quantity are now only found in Arctic and Antarctic waters" (l.c.), we know, that even now enormous invasions of diatoms occur from time to time on the Californian coast, as Prof. BAAS BECKING of Stanford University informed me.

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[1]) BRUCE L. CLARK, The marine tertiary of the West coast of the United States, etc. J. of Geol. **29**. 1921. p. 583—614.

[2]) When this article was already under the press I was informed by my friend Dr. W. HOTZ of Bâsle that diatomaceous deposits have already been described from NW Peru by A. WERENFELS (Eclogae geologicae Helvetiae **19**, 1926, 630—631 and **20**, 1927, 473—486). I am very sorry that I have not had under the eyes these essays, the „Eclogae" not being present in any library in Utrecht.

[3]) BRUCE L. CLARK, l.c., J. C. BRANNER, Bull. Geol. Soc. of America, **24**. 1913. p. 94.

When considering how quickly our views of the genesis of the marine diatomaceous depositions had to be reconsidered in consequence of new geological and biological facts, we have to ask ourselves whether perhaps in the future also our conception of the genesis of other sediments, notably of the radiolarian deposits, will have to be revised.

Palaeontology.

Operculina nummulitiformis n.s. (Fig. 1—10 and Pl. I, Fig. 11, 12).

Occurrence: Locality 130, South of Punta Pisura, D. 10604.
Locality 110, Paita, D. 10523.
Locality 78, N. W. of Vichayal, D. 10496, 10497.
Locality 60, Quebrada Punta brava, D. 10495.
Locality 45, Quebrada negra and Quebrada N. W. of Conchudo, D. 10488—10492.
Locality 44, Quebrada seca, D. 10480—10488, 10493, 10494.

Probably everywhere in eocene rocks.

Small, flat, nummulinids, entirely or for the greatest part involute, of which it is difficult to say to which genus they belong, to Operculina or to Nummulites. They possess 4—5 convolutions; in a few cases there is a beginning of a sixth convolution. The horizontal diameter is mostly less than 3 mm; in some ten specimens it amounted to 1.9—3.7 mm. The thickness (height) is 0.35—0.45 mm. The initial chamber has a diameter of from 45—100 μ; it is impossible to distinguish micro- and megalospherical forms. In the last whorl there are 26—32 chamberlets. The horizontal-section is rather nummulitiform than operculiniform: the vertical section is just the reverse.

It is well known that the large Nummulites, so typical of the European and the Asiatic Eogene have not yet been described from America. However, in the course of years a number of old-tertiary forms from this part of the world have become known, that are to be referred to the genus Operculina or to Nummulites. TH. WAYLAND VAUGHAN [1]) has given us a review of what we know of them. According to him by far most Nummulites, described by earlier authors from America, are in reality Operculinae, and only few, very small true Nummulites are known (?Middle Eocene to Oligocene), of which only N. parvula Cushman has received a proper description. Representatives of the genus Operculina are much more numerous: of fossil forms VAUGHAN (l.c.) and CUSHMAN [2]) report about ten, mostly well-described species. The Peruvian form differs from all these species in that it is small and exhibits many convolutions. As late as 1910 CUSHMAN [3]) insisted on it that Operculinae should have at most 4 whorls. Afterwards (l.c., 1921), however, he described a fossil form from the eocene Ocala-lime of Florida (Operculina Willcoxi Heilprin) that has about 5 whorls, which completely invest each other. This species has no doubt strong affinities with the

[1]) T. WAYLAND VAUGHAN, American and European larger Foraminifera. Bull. Geol. Soc. America. **35**. 1924. p. 785—822.

[2]) J. A. CUSHMAN, American Species of Operculina and Heterostegina. U. States Geol. Survey. Prof. Paper 128E. 1921.

[3]) J. A. CUSHMAN, A monograph of the foraminifera of the N. Pacific. U. States Nat. Museum. Bulletin **71**. IV. 1910. p. 36.

Peruvian species, but it is larger (according to the pictures up to 8 mm), and has in the last whorl more chambers (35—45). With Op. Willcoxi as well as with Op. nummulitiformis we have to do with an intermediate form between the genera Operculina and

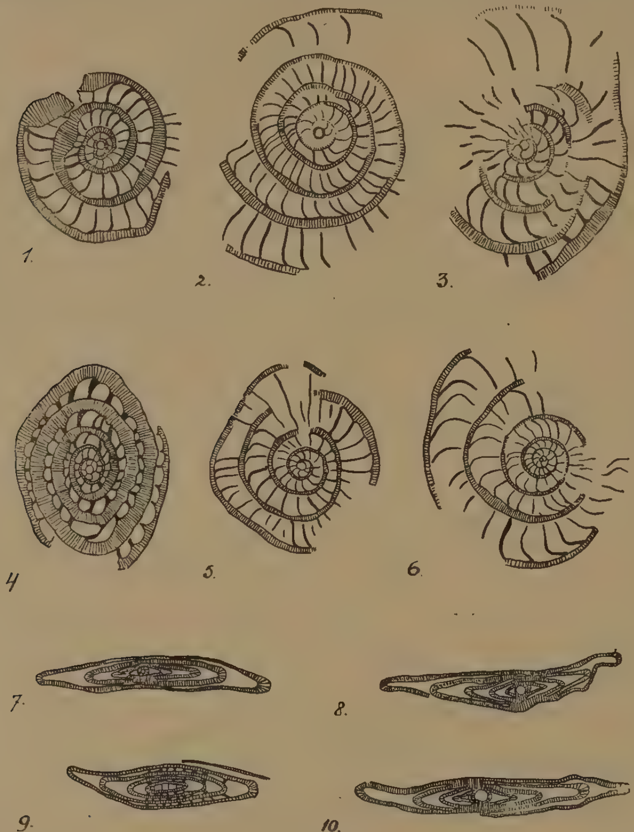


Fig. 1—5, 7—10. Magn. × 15; Fig. 6. Magn. × 10.
Fig. 1, 5 from D 10480; 8, 9, from D 10481; 6 from D 10484; 3, 4, from D 10487; 7, 10 from D 10493; 2 from D 10494.

Nummulites. Also CUSHMAN hesitated whether he should class Op. Willcoxi as an Operculina or a Nummulite (l.c. 1921).

It is probable, but not quite certain, that all specimens of Op. nummulitiformis have originated from eocene strata. The fossil is well typified and easy to recognize; it will possibly serve as an excellent guiding fossil.

Lepidocyclina (Polylepidina) variabilis n.s. (Pl. I, Fig. 13—22; further Fig. 23*a*—*h* and Fig. 24*a*—*e*).

Occurrence: Locality 133, East of Quitir, D. 10508—10521.
Locality 136, Punta Nonura.

Probably in eocene strata.

Very well-defined form. There are large, microspherical individuals, with a diameter of from 15—18 mm and a thickness of 2½ mm, and smaller megalospherical individuals from 5—8 mm in diameter. When full-grown they display both a very typical, swollen outer-edge, where the number of layers of median chambers is enlarged, and the lateral chambers are lacking (Fig. 16 and 17 on Pl. I). The fossils are most often circular with a more or less distinct tubercle in the centre; in normal condition they are developed in one plane (Pl. I, Fig. 19, 20 and 17). A few, however, are irregularly saddle-shaped (Pl. I, Fig. 18), no doubt in consequence of accommodation to abnormal conditions of growth. The surface exhibits numberless, very fine columns (Pl. I, Fig. 19, 20), arranged more or less concentrically. The embryonic chamber in the microspherical forms must be very small; I have not succeeded in visualizing it in a preparation (Fig. 21, 22 on Pl. I). The median chambers are very different as to size and shape (Fig. 24*a*—*e*). The microspherical forms have in their fullgrown stage in the central part 15—20 layers of lateral rooms on either side of the median-chamber-layer. The embryonic apparatus is very remarkable (Fig. 23*a*—*h*; Pl. I, Fig. 13, 14, 15). Its exceeding variability in shape

Fig. 23 *a*—*h*. Magn. × 20. Embryonal chambers of Polylepidina variabilis. *a* from D 10514; *b* from D 10510; *c* from D 10513; *d* from D 10515; *e* from D 10512; *f* from D 10520; *g* from D 10511; *h* from D 10521.

and in size has given origin to the name of the species. A single individual (Fig. 23*f*) displayed an Isolepidina-like embr. app.; in all the others the embr. app. consisted of more than 2 chambers, while the number may rise to 11, and no regularity whatsoever can be distinguished in the arrangement of the chambers.

VAUGHAN (l.c.) has devised the term Polylepidina for Lepidocyclines with an embr. app., that consists of several chambers not differing much in size, and has observed that the number of embryonal chambers can rise to 5. We may refer the fossils of Peru to this subgenus, provided that the definition includes also such cases in which the number of embr. chambers amounts to more than 5. The Peruvian fórms differ from the three Polylepidines described by VAUGHAN by the larger dimensions, by the very typical rim and by the incredible variability of the embryonic apparatus.

It is almost certain that the forms of Pol. variabilis have been found in eocene strata; if they occur more frequently in Peru, they will form excellent guides, as a large vertical distribution of such a specialized form is not imaginable. The Polylepidines described by VAUGHAN are also eocene.

Lepidocyclina (Isolepidina) R. Douvillei Lisson, var. armata nov. var. (Fig. 24, 1, m; Pl. II, Fig. 27, 28; Fig. 29*a*—*d*).

Occurrence: Locality 42, W. of Cerro Pinal, D. 10591, 10597—10599.
Locality 44, Quebrada seca.
Locality 69, Los Organos, D. 10600—10602.

At all localities in eocene rocks.

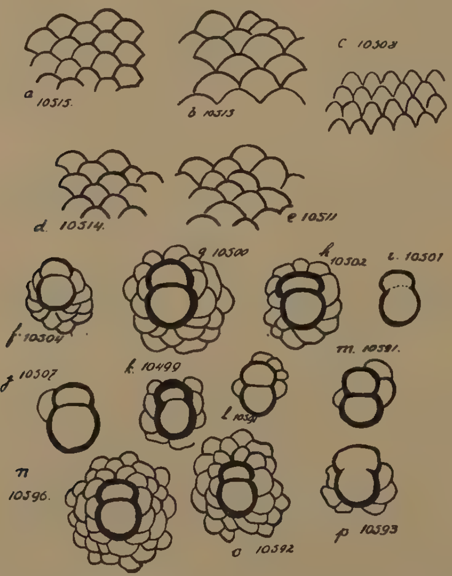


Fig. 24 *a*—*e* Polylepidina variabilis; *l*, *m*, Isolepidina R. Douvillei var. armata; *f*—*k*, *n*—*p*, Lepidocyclina (?Isolepidina) Vichayalensis. Magnif. circa × 32. *f* from D 10504; *g* from D 10500; *h* from D 10502; *i* from D 10501; *j* from D 10507; *k* from D 10499; *l* from D 10591; *m* from D 10591; *n* from D 10596; *o* from D 10592; *p* from D 10593.

To this variety a number of small Lepidocyclines have been referred, which, as to their main characteristics, agree with the species L. (Isolepidina) R. Douvillei, as described by LISSON [1]). The diameter of the megalospherical individuals is about 1½—2½ mm ; their thickness about 0.7—1 mm. The embryonal apparatus is typically isolepidine ; its largest width is about 250 μ (Fig. 24 *l, m*). The forms differ, however, from Lep. R. Douvillei in the fact that they always possess distinct and numerous columns (Fig. 27, 29*a*—*d*), which are largest and most numerous in the centre. This is why I have had to class them under a new variety. A few microspherical individuals have a diameter of up to 4 mm (Pl. II, Fig. 28).

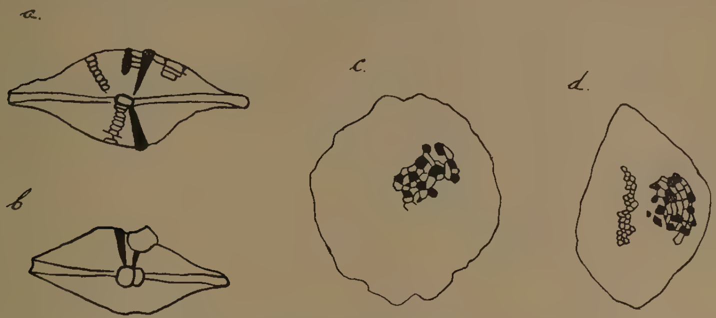


Fig. 29. Lepidocyclina R. Douvillei var. armata. *a, b* from D 10598, *c* from D 10597, *d* from D 10591.

H. DOUVILLÉ [2]) maintains that Lepidocyclina R. Douvillei is perhaps equivalent to Lep. Trinitatis. A. TOBLER has already pointed out that this cannot be right, since Lep. Trinitatis has numerous strong columns, which are wanting in Lep. R. Douvillei. Now the conception would be admissible that the "columned" variety of Lep. R. Douvillei is, in strictness, nothing but Lep. Trinitatis. But this is not the case : according to the dimensions given by H. DOUVILLÉ the megalospherical form of Lep. Trinitatis is much larger, and the embryonal apparatus is also very much larger. The diameter of the individuals is recorded 3—4 mm the largest diameter of the embryonal apparatus 0.5 mm.

Lepidocyclina (?Isolepidina) Vichayalensis n.s. (Fig. 24, *f*—*k, n*—*p*, Pl. II, Fig. 25, 26).

Occurrence : Locality 79, N. W. of Vichayal, (D. 10498—10507, 10592—10596).

The fossils were found in eocene Pariñas-sandstone.

These fossils are also allied to Lepidocyclina R. Douvillei, with which they agree especially in the lack of columns (Plate II, Fig. 26). But they differ from the species in question first of all by decidedly larger dimensions (the microspherical individuals have a diameter of 3—5 mm) and secondly by the structure of the embryonal apparatus.

[1]) C. LISSON, l.c.

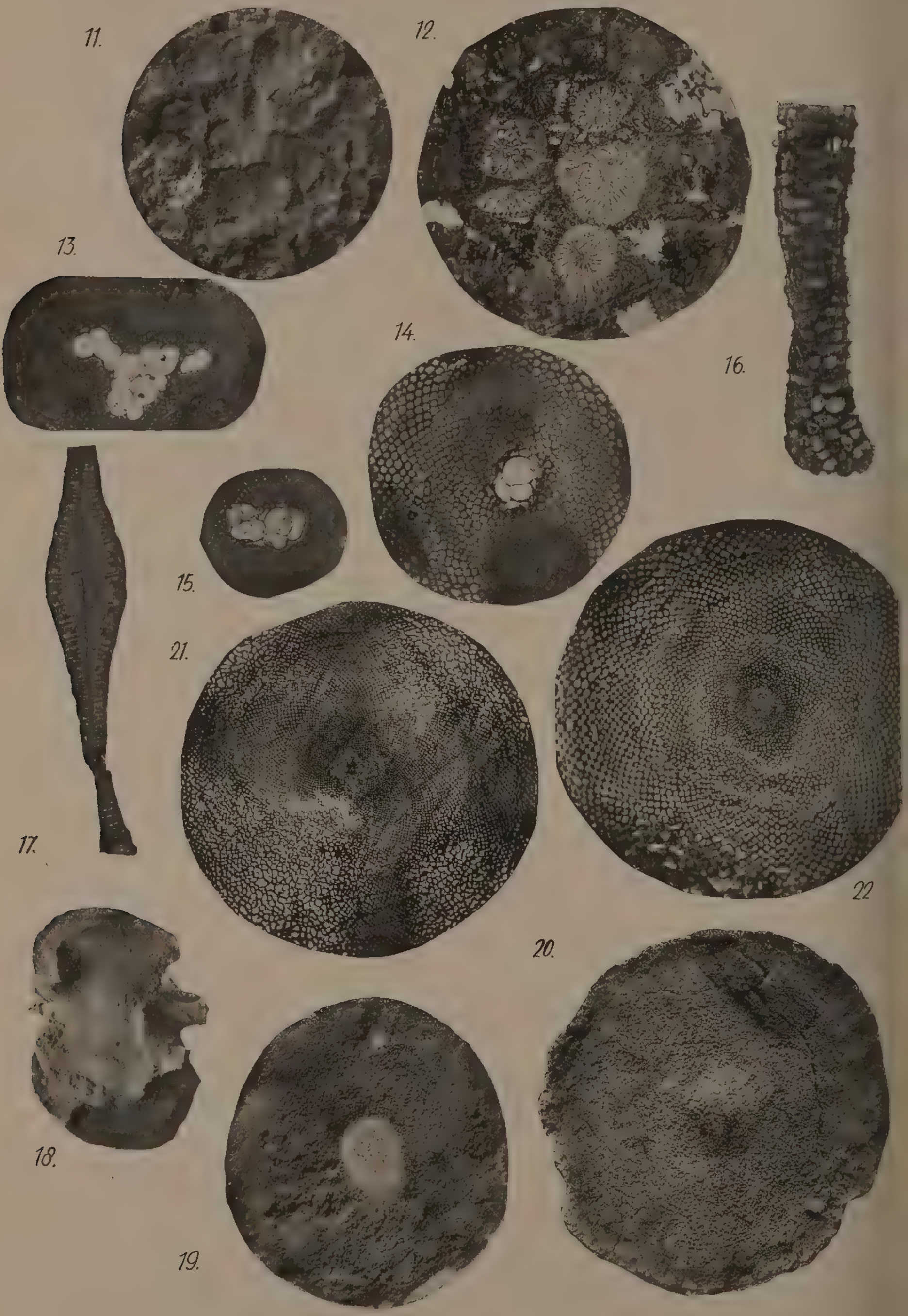
[2]) H. DOUVILLÉ, Mém. Soc. Géol. de France. Nouvelle série. I. 2, 1924. p. 34—36

Whereas in Lep. R. Douvillei this is strictly "isolepidine", in the fossils of Vichayal it regularly occurs that the septum, separating the first and the second chamber, is not straight but distinctly bent (Fig. 24 and 25). The embryonal apparatus — although resembling that of Isolepidina — deviates a little from it in the direction of Nephrolepidina. It was necessary, therefore, to devise a new species for these fossils, which I assigned to the subgenus Isolepidina, though I query its accuracy. The arrangement of the median chambers, in some individuals reminds us of Helicolepidina TOBLER.

EXPLANATION OF THE PLATES.

Pl. I, Fig. 11, 12 Operculina nummulitiformis. Fig. 11 magn. × 4, Fig. 12 magn. × 6.
Fig. 13—22 Lepidocyclina (Polylepidina) variabilis n.s.
Fig. 13, magn. × 10 from D 10513.
Fig. 14, magn. × 14 from D 10514.
Fig. 15, magn. × 11 from D 10510.
Fig. 16, magn. × 19 from D 10517.
Fig. 17, magn. × 6 from D 10518.
Fig. 18, magn. × 3½.
Fig. 19, magn. × 3½.
Fig. 20, magn. × 3½.
Fig. 21, magn. × 9 from D 10508.
Fig. 22, magn. × 9 from D 10515.

Pl. II, Fig. 25, 26, Lepidocyclina Vichayalensis n.s.
Fig. 25, magn. × 45 from D 10496.
Fig. 26, magn. × 45 from D 10494.
Fig. 27, 28 Lepidocyclina (Isolepidina) R. Douvillei Lisson, var. armata nov. var.
Fig. 27, magn. × 20.
Fig. 28, magn. × 25 from D 10599.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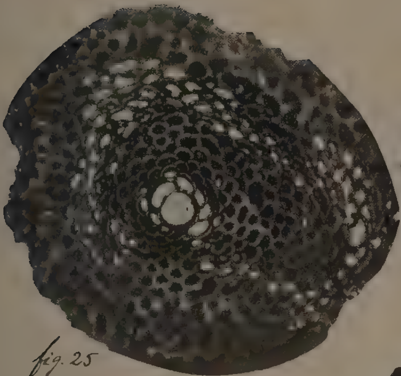

fig. 25

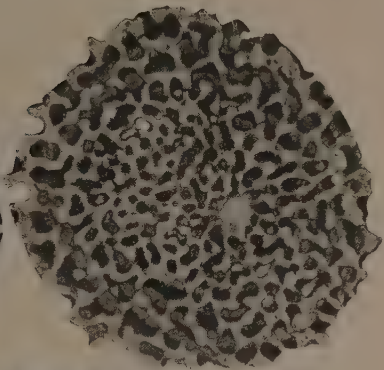

fig. 26.

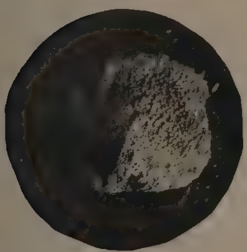

fig. 27

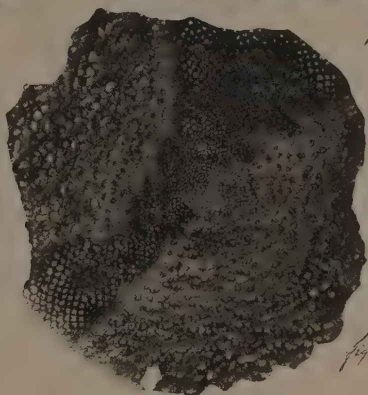

fig. 28

Physiology. — *District formation, mixing or fusion of myotomic derivates in dimeric mammalian muscles? Uni- or biradicular innervation?* [1]) By G. VAN RIJNBERK and L. KAISER.

(Communicated at the meeting of September 29, 1928).

Introduction.

Whether muscle fibres from different myotomes remain separated in the muscle, or, on the other hand, become mixed or even amalgamated, this is already an old problem.

Usually it has been tried to solve this question by investigating the way of innervation taken by the motor nerve fibres springing from the ventral spinal roots. Concerning conditions in mammals the following data are laid down in literature.

KRAUSE (1865) [2]) cut one innervating root at a time and determined microscopically the distribution of degenerated nerve fibres in the muscle. He concluded that each pluriradicular and therefore plurisegmental muscle consists of more or less distinct neuromuscular radicular territories. Each muscle fibre from such a territory therefore receives nerve fibres from one spinal root only; the radicular (segmental) territories remain separated.

FORGUE and LANNEGRACE (1885) [3]) believed to have demonstrated that stimulating one of the spinal roots of a plurisegmental muscle results in contraction of the whole muscle. Therefore they concluded that district formation (so-called "cantonnement"), i.g. division in separate radicular territories, does not exist.

EXNER (1885) [4]) failed to find degeneration in the cricothyreoid muscle of the rabbit after cutting the N. Laryngeus sup. and med.; also the

[1]) After research carried out in the Physiological Laboratory of the University of Amsterdam.

[2]) W. KRAUSE. Beitr. zur Neurologie der oberen Extremität. 1865.

[3]) E. FORGUE et W. LANNEGRACE. Distribution des racines motrices dans les muscles du membre supérieur, etc. Comptes rendus de l'Académie des Sciences. Tome **98**. p. 685, 829, 1068.

[4]) S. EXNER. Notiz zu der Frage von der Faservertheilung mehrerer Nerven in einem Muskel. Pflüger's Archiv. **36**. S. 572.

gastrocnemius muscle did not show any degeneration after cutting the larger of the three plexus stems from which the muscle receives its innervation.

SHERRINGTON (1892)[1]) and VAN RIJNBERK (1911)[2]) on the other hand confirmed KRAUSE's opinion by means of electrical stimulation of the spinal roots; each root innervates a seperate district of the pluri-radicular muscle in the innervation of which it partakes. However the remark must be made that both investigators worked with long muscles consisting of segmental or radicular parts one situated after the other, i.g. the M. Sartorius (Sh.) and the M. Iliocost. (V.R.)

LEDERER and LEMBERGER (1907)[3]) tried to solve the contradictory results obtained by EXNER with rabbit and frog and by GAD[4]) (1882) with frogs, by mammal experiments, in which the cricothyreoid muscle of the rabbit was stimulated. Those experiments yielded the conclusion that part of the muscle fibres received their innervation from one innervating nerve, partly from the other. Analogous experiments with the M. Flexor digt. comm. prof. and subl., both innervated by C VIII and Th I gave opposite results; it appeared that all or at least the majority of muscle fibres are innervated by nerve fibres originating from both roots.

After those researches the question remained untouched for some time till E. AGDUHR[5]) took it up again in 1917—19.

The outcome of stimulating the median and ulnar nerve of the pig gave him the impression that the muscle fibres of the M. Flexor digit. comm. receive a double innervation at the same time from median fibres (C VIII) and from ulnar fibres (Th. I). His extensive anatomical control work on the cat, by means of the degeneration method, confirmed this conclusion.

More recent investigators worked with frogs exclusively. (We mention FULTON 1925, DE BOER 1925, SAMOILOFF 1926). None of those could confirm the results of AGDUHR; all found that muscle fibres of polymer

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1) C. S. SHERRINGTON. Notes on the arrangement of some motor fibres in the lumbo-sacral plexus. Journal of Physiology. Vol. **13**, p. 621.

2) G. VAN RIJNBERK. Ueber die Segmental-Innervation polymerer Muskeln. Ein Beitrag zur Cantonnementfrage. Folia Neuro-biologica. V. **7**. blz. 767.

3) R. LEDERER und F. LEMBERGER. Zur Frage der doppelten Innervation von Muskeln des Warmblüters. Pflüger's Archiv. **119**. S. 95.

4) J. GAD. Ueber einige Beziehungen zwischen Nerv, Muskel und Zentrum. Festschrift z. Feier des 300-jahrigen Bestehens der Julius Max. Univ. zu Würzburg.

5) E. AGDUHR. Anatomische, statistische und experimentelle Untersuchungen über den N. ulnaris und N. medianus. Anat. Hefte. **52**. H. 158.

Morphologischer Beweis der doppelten (plurisegmentalen) motorischen Innervation der einzelnen quergestreiften Muskelfasern bei den Säugetieren. Anat. Anzeiger. **49**. S. 1.

Ueber die plurisegmentelle Innervation der einzelnen quer-gestreiften Muskelfasern. Anat. Anzeiger. **52**. S. 273.

muscles of the frog are innervated by fibres from one spinal root only, an opinion that was already forwarded by GAD in 1882, therefore no amalgamation of contractile substance originating from different myotoms takes place.

As yet it appears that the problem is by no means solved.

EXPERIMENTAL CONTRIBUTION.

In our inquiry into the segmental structure and radicular innervation of the M. Rectus abdominis of the dog we found that very frequently one and the same muscle district was supplied with nerve fibres from two different roots. Those muscle segments were designated in our terminology as M 4 and M 5, clearly separated by tendinous septa (myosepta, inscriptiones tendinea), and received their innervation from Th 11 and 12 in the first case, from Th 12 and 13 in the second case. We have tried to solve the question of district formation ("cantonnement") by means of such dimeric muscle segments. Those segments consist of muscle fibres of paralel course, have a very simple form (about that of a paralellopipedum) and are easily accessible for inspection and experiment. We used to study the results of stimulating the spinal roots or of the unisegmental peripheral nerve branches.

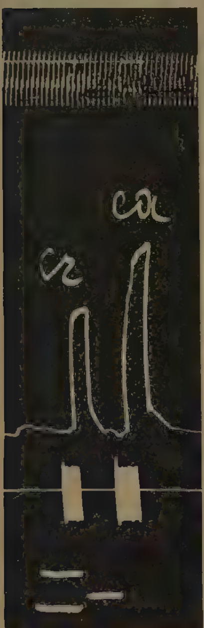

Fig. 1.

1. *Data obtained by simple observation of the contractions of muscle segments innervated by two roots.*

The general result of root or nerve stimulation is increase in bulk and shortening in cranio-caudal direction. But if first the cranial and afterwards the caudal root or segmental branch of the double innervated muscle segment be stimulated and the changes in form are carefully noted, it sometimes will become apparent that the change in both cases is absolutely the same. It is impossible to deduct from the character of deformation of the muscle segment, which of the two roots has been stimulated. Therefore in this case, formation of district does not exist, and amalgamation is possible. But in other cases the deformation of the contracting muscle segment by stimulating one or the other root is quite different. Often the contraction elicited from the cranial nerve is stronger on the medial side. Such facts prove without doubt that in double innervated muscle segments, district formation may occur. But whether in the cases just mentioned a double innervation of the muscle fibres exists is not to be answered by simple observation; if a complete mixing

of simple innervated fibres has taken place without amalgamation, consecutive stimulation of both roots should yield the same result.

Fig. 2.

2. Data obtained by graphical method; registration of change in diameter of muscles innervated by two roots.

In order to answer the question whether the fibres of a muscle innervated by two roots receive a single (uniradicular) or double (biracular) innervation, several procedures are indicated, some of which have been applied by us.

We have registered the change in diameter of a dimeric muscle segment by inserting the part between the legs of a pair of MAREY muscle pincers, the movement being registrated on a smoked drum by means of a tambour worked by air transport. The flat shape of the rectus muscle is well adepted to those experiments. The results are as follows.

A. The tetanic contraction obtained by maximal faradic stimulation of the separate roots or nerves were never of aequal height. (Fig. 1). By

stimulattng one root the apex of contraction is always heigher than by stimulating the other. If amalgamation existed in the derivates of myomers in those muscle segments and the fibres of it receive a double (biradicular) innervation, this fusion could not have been complete, not all muscle fibres are double innervated, for it is clear that one root supplies more fibres than the other.

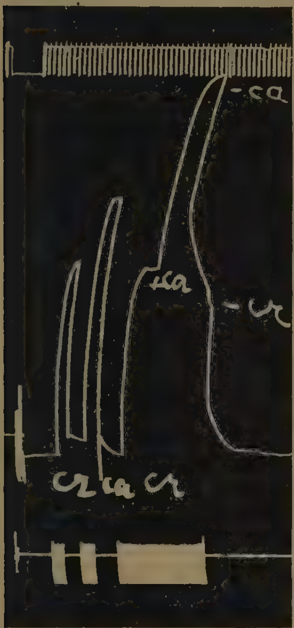

Fig. 3.

B. When one of the roots or nerves was stimulated for a certain period at a time till the height attained decreased for about $^2/_3$ and the muscle therefore was fatigued, the height would increase again immediately to the normal niveau when the other root or nerve was stimulated. (Fig. 2). This shows that at least in the place where the curve was registrated, each root supplied different muscle fibres and that for certain no considerable number of double innervated fibres could be present. Changing the position of the myograph (median, in the middle, lateral) always gave the same result.

C. When first one root was stimulated for a short time and than the other root or nerve together with the first (also for a short time to avoid fatigue) the height of contraction would increase considerably (Fig. 3) This also proves clearly that muscle fibres innervated by one root are not the same as those innervated by the other root, or at least, that the majority of the fibres receive an uniradicular innervation.

Summary of experimental results.

In some cases biradicular innervated segments of the rectus abdominis muscle show division in districts ("cantonnement"); the medullar roots innervating the muscle parts each supply a distinct territory with fibres. The large majority of the muscle fibres receives in those cases uniradicular innervation.

2. In other cases both spinal roots distribute their fibres throughout the muscle segment. But double innervation does no more exist than in the former case; the majority of muscle fibres receives its innervation from one single medullar root.

Conclusion.

Sometimes the muscle segments of the M. Rectus abdominis of the dog

that are innervated by two spinal roots show district formation; sometimes the territories of the roots are completely mixed. But each fibre is innervated by one root only (uniradicular). If biradicular innervation be present, it occurs as an exception. If those data be applied to the segmental origin of muscle, the following conclusion is justified: sometimes in the rectus abdominal muscle of the dog the contractile substance originating from two myomers remains partly separated (district formation), sometimes intensive mixing may take place. The experimental results do not allow to assume the occurrence of amalgamation of contractile material.

EXPLANATION OF THE FIGURES.

Fig. 1. Tetanic stimulation of short duration, first of the cranial than of the caudal nerve supplying M 4, with the same faradic current. Time in seconds.

Fig. 2. Tetanic stimulation of long duration of caudal nerve supplying M 4. After marked fatigue of the muscle the POHL's switch is turned, enabling the stimulation of the cranial nerve with the same current. Time in seconds.

Fig. 3. Consecutive stimulation of short duration with the same current of cranial and caudal nerve separately. After this a series of stimulation: *A.* of cranial nerve only; *B.* of both nerves simultaniously; *C.* of cranial nerve only. Time in seconds.

Mathematics. — *Zur Geschichtschreibung der Dimensionstheorie.* By Prof. L. E. J. BROUWER.

(Communicated at the meeting of September 29, 1928).

§ 1.

Die Dimensionstheorie habe ich in einem 1913 erschienenen Aufsatz „Ueber den natuerlichen Dimensionsbegriff" (Journ. f. Math. 142, S. 146—152) fuer diejenigen Spezies, fuer welche die Dimension einen natuerlichen Sinn besitzt, naemlich fuer die kondensierten metrischen Spezies, begruendet, indem ich einen dimensionellen Rechtfertigungssatz brachte, d.h. einen im Bereich der kondensierten metrischen Spezies sinnvollen Dimensionsbegriff definierte, fuer den ich zeigen konnte, dass die n-dimensionalen Mannigfaltigkeiten die Dimension n besitzen.

Fuer diesen Dimensionsbegriff boten sich naturgemaess zwei Varianten dar, welche den beiden damals fuer kondensierte metrische Spezies gangbaren Zusammenhangsbegriffen entsprachen. Es fuehrt naemlich in gleicher Weise der schwache Zusammenhang, d.h. die Unzerlegbarkeit in zwei Relativgebietspezies (LENNES 1911), zur starken Trennung (LENNES 1911), zur schwachen Dimension und zum schwachen Rechtfertigungssatz, wie der starke Zusammenhang, d.h. die Unzerlegbarkeit in zwei Stueckspezies (SCHOENFLIES 1908), zur schwachen Trennung, zur starken Dimension und zum starken Rechtfertigungssatz. Der schwache Rechtfertigungssatz ist eine unmittelbare Folge des starken, aber nicht umgekehrt. Ein Beweis des (im unten besprochenen MENGERschen Buche nicht erwaehnten) starken Rechtfertigungssatzes wurde erst 1924 veroeffentlicht und zwar in meiner Note „Bemerkungen zum natuerlichen Dimensionsbegriff" (diese Proceedings 27, S. 635—638).

In meinem zitierten Aufsatz vom Jahre 1913 wird zwar schon der starke Rechtfertigungssatz dem Wortlaute nach formuliert, der darauf folgende Beweis bezieht sich aber aufs deutlichste auf den schwachen Rechtfertigungssatz, so dass ersichtlich nur dieser gemeint ist, und der in der Formulierung des Satzes enthaltene Schreibfehler jedem mitdenkenden Leser sofort offenbar wird. Dass dieser Schreibfehler lange Jahre hindurch keine gedruckte Verbesserung erfuhr, hatte erstens die allgemeine Ursache der Fahrlaessigkeit muehelos einzusehenden Schreibfehlern gegenueber, zweitens einen in diesen Proceedings 27, S. 136 mitgeteilten speziellen Grund.

In meinem zitierten Aufsatz von 1913 habe ich mich auf die Begruendung der Dimensionstheorie beschraenkt und auf die Veroeffentlichung weiterer dimensionstheoretischer Entwicklungen verzichtet,

einerseits weil mit dem Beweis des Rechtfertigungssatzes das gestellte erkenntnistheoretische Ziel erreicht war, andererseits weil [1]) fuer die anschliessenden Betrachtungen (in erster Linie diejenigen welche sich um den Summensatz und den Zerlegungssatz gruppieren) nicht wie fuer den Beweis der Rechtfertigungssatzes [2]) eine intuitionistische Verwirklichung nahe lag.

Die Aufstellung des schwachen Dimensionsbegriffes und der Beweis des schwachen Rechtfertigungssatzes, d. h. die Begruendung der Dimensionstheorie, ist mit meiner zitierten Arbeit von 1913, und nicht etwa mit der ersten oeffentlichen Formulierung des in derselben enthaltenen offensichtlichen Versehens, ebenso unanfechtbar zu datieren, wie der Beweis der Invarianz des Gebietes mit meiner betreffenden Arbeit von 1911, und nicht mit der ersten oeffentlichen Ausfuellung der in der ihr zugrunde liegenden Definition der Pseudomannigfaltigkeit befindlichen offensichtlichen Luecke, oder auch die mengentheoretische Begruendung der ebenen Geometrie mit HILBERT's bezueglichem Originalaufsatz von 1906, und nicht mit dem Wiederabdruck dieses Aufsatzes, in welchem meinen brieflichen Angaben vom Oktober 1909 zufolge ein offensichtliches Versehen in den zugrunde liegenden Axiomen richtiggestellt wurde.

§ 2.

Auf meine im § 1 besprochene Begruendung der Dimensionstheorie folgen URYSOHN, der 1922, von einer lokalen Form der schwachen Dimensionsdefinition [3]) ausgehend, den Summensatz und den Zerlegungssatz fand [4]) und im Zusammenhang mit diesen Saetzen in den Jahren 1922—1924 zahlreiche dimensionstheoretische Fragen zur Klaerung brachte [5]), und MENGER und ALEXANDROFF, die, ersterer seit 1924, letzterer seit 1925, die Dimensionstheorie weiter ausbauten und vertieften. Weil indessen die scharfsinnigen Untersuchungen von URYSOHN, MENGER und ALEXANDROFF grossenteils einen bloss formalistischen Charakter besitzen, reicht der intuitionistische (d.h. sinnvolle) Bestand der Dimensionstheorie heute kaum ueber denjenigen von 1913 hinaus.

§ 3.

Im neulich erschienenen MENGERschen Buche "Dimensionstheorie" wird eine historische Darstellung der Entstehung der Dimensionstheorie

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[1]) Vgl. meine einschlaegige Aeusserung in „Intuitionistische Mengenlehre", diese Proceedings 23 (1920), S. 950.

[2]) Vgl. meine (im unten besprochenen MENGERschen Buche nicht erwaehnte) Note „Intuitionistische Einfuehrung des Dimensionsbegriffes", diese Proceedings 29 (1926), S. 855—863.

[3]) Auf die (im unten besprochenen MENGERschen Buche nicht explizite erwaehnte) Aequivalenz der URYSOHNschen Form mit der meinigen haben URYSOHN fuer kompakte Spezies und ich fuer kondensierte Spezies unabhaengig voneinander hingewiesen (vgl. Fund. Math. 8, S. 328; diese Proceedings 27 (1924), S. 636).

[4]) Comptes Rendus 175, S. 440.

[5]) Fund. Math. 7, 8.

gegeben, welcher gemaess MENGER als ihr Begruender, ich (mit EUKLID und einigen anderen) als Vorlaeufer und URYSOHN als Nachfolger zu gelten haetten. Diese Darstellung widerspricht dem oben in § 1 und § 2 dargelegten wirklichen Sachverhalt, ist denn auch falsch und bedeutet ein Antasten der Autorschaft eines Lehrers und eines verstorbenen Kommilitonen.

§ 4.

Meine Autorschaft wird im MENGERschen Buche angetastet dadurch, dass zu verstehen gegeben wird, dass die BROUWERsche Note aus dem Jahre 1913 eine Definition der schwachen Dimension (jedenfalls nach dem damaligen Stande der Wissenschaft) weder explizite noch implizite enthalte, dass also derjenige, der spaeter eine solche Definition explizite niedergeschrieben habe, berechtigt sei, sich selbst als Entdecker des Dimensionsbegriffes und als Begruender der Dimensionstheorie zu qualifizieren. Diese These wird erhaertet mit zwei falschen Zitaten:

Erstens wird (auf S. 85) meine Arbeit von 1913, die in den bisherigen Veroeffentlichungen MENGERs regelmaessig regulaer mit der schwachen Dimensionsdefinition, d.h. unter Anbringung der aus dem anschliessenden Text folgenden, zum Ueberflusse auch formal und explizite in der Literatur befindlichen Richtigstellung, zitiert worden war, hier auf einmal mit der starken Dimensionsdefinition angefuehrt, wobei also die MENGER einerseits als sich unmittelbar darbietend, andererseits als in der Literatur vorhanden bekannte Richtigstellung in einer den Leser irrefuehrenden und den Sachverhalt entstellenden Weise totgeschwiegen wird.

Zweitens wird (auf S. 86) im Anschluss an das vorstehende Zitat behauptet, dass die der daselbst dem Wortlaute nach enthaltenen Form der starken Definition dem Wortlaute nach aehnliche Form der schwachen Definition auf dem „inzwischen erst entstandenen" modernen Zusammenhangsbegriff beruhe (eine Behauptung, der man kaum einen anderen Sinn beilegen kann, als den einer Anfechtung der frueheren — sowohl durch mich wie durch MENGER erfolgten — Hinstellung des betreffenden Versehens als Schreibfehler), waehrend in Wirklichkeit die Definition des schwachen Zusammenhangs (wie diejenige der starken Trennung) in einer von MENGER selbst in seiner ersten Veroeffentlichung ordnungsgemaess zitierten Abhandlung von LENNES aus dem Jahre 1911 vorzufinden ist.

§ 5.

Die URYSOHNsche Autorschaft in bezug auf die lokale Form der schwachen Dimensionsdefinition wird im MENGERschen Buche angetastet mit der Anfuehrung (daselbst S. 83 u. 84) von zwei Akten, welche der zitierten URYSOHNschen Comptes-Rendus-Note gegenueber eine Prioritaet besitzen sollen, naemlich:

1. Eine Hinterlegung bei der Wiener Akademie der Wissenschaften aus dem Jahre 1921.

Diese (mir vorliegende, uebrigens in diesen Proceedings 29, S. 1122—1123 wörtlich abgedruckte) Hinterlegung enthaelt aber nur vage Andeutungen ueber die Dimensionalitaet „gewisser Kontinua" der euklidischen Raeume, denen kaum mehr als die Hoffnung, dieselbe spaeter einmal durch eine rekursive Definition festzulegen, zu entnehmen ist (Schon um die Dimension 2 zu definieren wird der dazu notwendige Begriff der Dimensionalitaet 1 fuer kompakte Mengen welche keine Kontinua sind, als „aus diskreten Kurven bestehend" erklaert, mit der nicht weiter ausgefuehrten Bemerkung, dass „dieses diskret sich praezisieren lasse").

2. Eine angeblich im Februar 1922 bei den Monatsh. f. Math. u. Phys. eingereichte Note, in welcher die schwache Dimensionsdefinition fuer Teilmengen von euklidischen Raeumen ausgesprochen sein soll.

In Wirklichkeit ist eine derartige Note nicht erschienen und es hat sich davon auch kein Manuskript vorgefunden. Es existiert nur ein (mir vorliegender) Brief von MENGER an HAHN vom 15. Febr. 1922, der folgenden Passus enthaelt: *„Ich hatte die kleine Arbeit, die Sie Herr Professor zu lesen die grosse Guete hatten, mit einer Definition der n-dimensionalen Mengen beschlossen*[6]*, die korrekterweise, wie ich nun glaube*[6]*, so haette lauten sollen:*" (Es folgt die lokale Form der schwachen Dimensionsdefinition fuer Teilmengen von euklidischen Raeumen). „..... *Ich vermute*[6]*) nun, dass u.a. folgender Satz gilt: n-dimensional im R_n sind die und nur die Mengen mit einem innern Punkt im R_n.*"

Es handelt sich also nur um eine Vermutung (nicht einmal um eine unbewiesene Behauptung) des schwachen Rechtfertigungssatzes bzw. der Brauchbarkeit der lokalen Form des schwachen Dimensionsbegriffes fuer Teilmengen des R_n, also keineswegs um eine Begruendung, geschweige denn um die erste Begruendung, der Dimensionstheorie.

Ebensowenig wie die beiden obigen Akten gibt MENGER's erste Publikation „Ueber die Dimensionalitaet von Punktmengen, Erster Teil" (der Druckerei ueberreicht 12. 12. 1923, druckfertig erklaert 28. 1. 1924, erschienen 12. 4. 1924, auf dem Umschlage der Monatsh. f. Math. u. Phys. 33 und im MENGERschen Buche „Dimensionstheorie" unrichtig datiert mit dem Erscheinungsjahre 1923) ihm einen Anteil an der Begruendung oder dem Ausbau der Dimensionstheorie, weil in dieser Publikation nichts steht, was am Datum ihrer Einreichung nicht laengst (naemlich in meiner Arbeit von 1913 und in der URYSOHNschen Comptes-Rendus-Note von 1922) veroeffentlicht und allgemein zugaenglich waere.

Die fuer Prioritaetsfragen in Betracht kommenden MENGERschen Veroeffentlichungen zur Dimensionstheorie setzen erst ein mit seinen

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[6]) Sperrung von mir.

Aufsaetzen „Ueber die Dimension von Punktmengen" (diese Proceedings 27, S. 639—643, 1924) und „Ueber die Dimension von Punktmengen, II. Teil" (Monatsh. f. Math. u. Phys. 34, S. 137—161, der Redaktion eingereicht 6. 10. 1924, der Druckerei ueberreicht 15. 10. 1924, druckfertig erklaert 30. 12. 1924, erschienen 20. 9. 1926, auf dem Umschlage der Monatsh. f. Math. u. Phys. 34 und im MENGERschen Buche „Dimensionstheorie" unrichtig datiert mit dem Erscheinungsjahre 1924). Bei MENGER's Erscheinen auf der dimensionstheoretischen Buehne lagen mithin in der Literatur bereits vor die Definition der schwachen und der starken Dimension (inklusive des Aequivalenzbeweises der URYSOHNschen Form mit der meinigen fuer die schwache Dimension), die Formulierung und der Beweis des schwachen und des starken Rechtfertigungssatzes (inklusive der — in MENGER's Debutarbeit ordnungsgemaess zitierten — oeffentlichen formalen Richtigstellung des oben in § 1 und § 4 erwaehnten Schreibfehlers) und die Formulierung der Hauptsaetze fuer die schwache Dimension.

§ 6.

Die URYSOHNsche Autorschaft in bezug auf die Beweise des Summensatzes und des Zerlegungssatzes wird im MENGERschen Buche angetastet dadurch, dass daselbst (S. 94 u. 157) die MENGERschen Beweise dieser Saetze mit 1924 und die URYSOHNschen [7]) mit 1926 datiert werden.

Der Sachverhalt ist aber dieser, dass die (im Fruehling 1923 eingereichte) einschlaegige URYSOHNsche Abhandlung sicher nicht spaeter als mit URYSOHN's Todestag am 17.8.1924 und die einschlaegige MENGERsche Abhandlung nach dem vorigen § sicher nicht frueher als mit ihrem Eingangstag am 6. 10. 1924 zu datieren ist, so dass auch in bezug auf die Beweise des Summensatzes und des Zerlegungssatzes URYSOHN's Prioritaet ausser Zweifel steht.

Ueberdies wird die URYSOHNsche Autorschaft in bezug auf die Formulierung und den Beweis der Umkehrung des Zerlegungssatzes im MENGERschen Buche verdunkelt dadurch, dass der Zerlegungssatz und seine Umkehrung daselbst (S. 156 u. 157) nicht als selbstaendige Saetze, sondern als Teile von durch Kombination mit Ergaenzungen geringerer Tragweite entstehenden erweiterten Saetzen eingefuehrt werden.

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[7]) Die URYSOHNsche Formulierung des Summensatzes befindet sich nicht, wie MENGER angibt, auf S. 260, sondern auf S. 337 von Fund. Math. 8.

Mathematics. — *A Representation of a Complex of Biquadratic Twisted Curves of the First Kind on Point Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of September 29, 1928)..

§ 1. The complex of the biquadratic twisted curves of the first kind k^4 that pass through five given points $A_1, \ldots, A_5$, cut a given line a_1 twice and a given line a_2 once, may be represented on the points of space in the following way. We choose a quadratic cone K^2 and a line c. We suppose a projective correspondence to be established between the points K of a_1 and the tangent planes $\varkappa$ of K^2 and another one between the points C of a_2 and the planes γ through c. To a curve k^4 that cuts a_1 in K_1 and K_2, and a_2 in C, we associate the point of intersection of the planes $\varkappa_1$, $\varkappa_2$ and γ associated to K_1, K_2 and C.

§ 2. *The vertex T of K^2 is a cardinal point; in T are represented the ∞^2 curves k^4 that pass through the point A of a_2 which is associated to the plane cT.*

c is a singular line; the ∞^2 curves k^4 that are represented on c, cut a_1 in pairs of points of a quadratic involution I.

We have still to investigate whether it is possible that of a group of eight associated points five lie in $A_1, \ldots, A_5$, two on a_1 and one on a_2; for the consequence would be the appearance of a cardinal point. This, however, is not the case. If in a group of eight associated points five lie in $A_1, \ldots, A_5$ and two on a_1, the eighth lies on the twisted cubic that passes through $A_1, \ldots, A_5$ and has a_1 as chord. As a rule this curve does not cut a_2.

§ 3. There are ∞^1 curves k^4 that are singular for the representation, viz. the curves k^4 that pass through A and cut a_1 in a pair of points of I. They are represented on the rays of the pencil in the plane cT that has T as centre and they lie on the quadratic surface through A, $A_1, \ldots, A_5$ that contains a_1.

§ 4. Our complex contains the following systems of ∞^1 degenerate curves k^4:

The twisted cubic through $A_1, \ldots, A_5$ that has a_1 as chord, is completed by its chords that cut a_2, to a system of degenerate curves k^4 that is represented on a line through T.

Each of the five twisted cubics that pass through $A_1, \dots, A_5$ and cut a_1 and a_2, is completed by its chords that cut a_1, to a system of degenerate curves k^4 that is represented on a line cutting c and touching K^2.

The twisted cubics that pass through $A_1, \dots, A_5$ and cut a_1, are completed by their chords which cut a_1 and a_2, to a system of degenerate curves k^4 that is represented on a biquadratic curve c^4 that passes through T and cuts c three times and which is, accordingly, of the second kind. We prove this by cutting c^4 by a tangent plane $\varkappa$ of K^2. The number of points of intersection outside T is equal to the number of curves of the system that pass through the point K of a_1 that is associated to $\varkappa$. Now through K there passes one twisted cubic that passes at the same time through $A_1, \dots, A_5$ and is completed by its two chords that cut a_1 and a_2 to degenerate curves k^4. Further there is one twisted cubic that passes through $A_1, \dots, A_5$, cuts a_1 outside K, and has a ray of the plane pencil (K, a_2) as chord. This is a consequence of the property that the twisted cubics which pass through five given points, produce a polar correspondence in an arbitrary plane, so that the three points of intersection of any of the curves with the plane form a polar triangle [1]). The aforesaid twisted cubic forms a degenerate curve k^4 through K with the ray of the plane pencil (K, a_2) which it has as chord. Accordingly $\varkappa$ cuts c^4 outside T in three points. From the above mentioned property it also follows that c^4 has a singular point in T and that a plane through c cuts c^4 outside c in one point, so that c^4 cuts c three times.

The twisted cubics that pass through $A_i, \dots, A_m$ and cut a_1 once and a_2 twice, are completed by their chords through A_n to a system of degenerate curves k^4 that is represented on a twisted cubic which passes through T and has c as chord. This is proved in a similar way as above.

A twisted cubic that passes through $A_i, \dots, A_m$ and cuts a_1 twice, cuts a ray of the plane pencil (A_n, a_2) only then twice when this ray lies with $A_i, \dots, A_m$ and a_1 on one quadratic surface. If S_1 and S_2 are the points where the plane $A_n a_2$ is cut outside A_n by the twisted cubic that passes through $A_1, \dots, A_5$ and has a_1 as chord, the quadratic surface through $A_i, \dots, A_m$ and a_1 and $A_n S_1$ (resp. $A_n S_2$) contains ∞^1 twisted cubics that pass through $A_i, \dots, A_m$ and cut a_1 and $A_n S_1$ (resp. $A_n S_2$) twice and that are completed by $A_n S_1$ (resp. $A_n S_2$) to a system of degenerate curves k^4 which is represented on a line cutting c.

The twisted cubics that pass through $A_i, \dots, A_m$, cut a_1 and a_2, and have a ray of the plane pencil (A_n, a_1) as chord, are completed by these chords to a system of degenerate curves k^4 that is represented on a curve of the order six that passes through T and cuts c five times.

The transversal through A_n of a_1 and a_2 is completed by the twisted

1) Vg. R. STURM, Die Lehre von den geometrischen Verwandtschaften, 4, 103.

cubics that have this transversal as chord, pass through $A_i, \dots, A_m$ and cut a_1 to a system of degenerate curves k^4 that is represented on a line which cuts c and touches K^2.

The line $A_i A_k$ is completed by the twisted cubics that pass through A_l, A_m, A_n, cut $A_i A_k$ and a_1 twice and a_2 once, and which, accordingly, lie on the quadratic surface that passes through A_l, A_m, A_n and contains $A_i A_k$ and a_1, to two systems of degenerate curves k^4 that are represented on two lines which cut c.

The conic that passes through A_i, A_k, A_l and cuts a_1 and a_2, is completed by the conics that pass through A_m, A_n and cut the said conic twice and a_1 once, to a system of degenerate curves k^4 that is represented on a line which cuts c and touches K^2.

The conics that pass through A_i, A_k, A_l and cut a_1, are completed by the conics that pass through A_m, A_n, cut a_1 and a_2 and cut one of the aforesaid conics twice, to a system of degenerate curves k^4 that is represented on a conic which passes through T, cuts c, and lies in a tangent plane of K^2.

§ 5. K^2 is the image surface of the system of the curves k^4 that touch a_1.

The curves k^4 that pass through a given point P, lie on the quadratic surface ω^2 that passes through $A_1, \dots, A_5$ and P and contains a_1. Let a_2 cut ω^2 in P_1 and P_2. The curves k^4 on ω^2 that pass through P and P_1 as well as those that pass through P and P_2 cut a_1 in pairs of points of a quadratic involution. Consequently the system of the curves k^4 that pass through a given point P, is represented on two lines a_P and a'_P, that cut c.

a_P and a'_P together cut K^2 in four points; hence:

There are four curves k^4 that pass through a sixth given point and touch a_1.

This number can also be deduced directly. For each of the above mentioned involutions on a_1 has two double points.

§ 6. Let k_b be the image curve of the system Σ_1 of the curves k^4 that have a given chord b. By making use of the property that the biquadratic curves of the first kind that pass through six given points and cut a given line twice, lie on the quadratic surface that is defined by these elements, we find that through a given point of a_1 or a_2 there pass two resp. one curve of Σ_1. Hence a tangent plane K^2 cuts k_b in all in three points. Accordingly k_b is a twisted cubic that passes through T and has c as chord.

k_b cuts K^2 outside T in four points. Hence:

There are four curves k^4 that touch a_1 and cut a third given line twice.

§ 7. Let us call O_l the image surface of the system Σ_2 of the curves k^4 that cut a given line l. In order to determine the degree we cut O_l by a line that cuts c and touches K^2. The number of points of intersection outside c is equal to the number of curves of Σ_2 that pass through a given point of a_1 and a given point of a_2. This number is equal to two as the biquadratic curves of the first kind that pass through seven given points and cut a given line, form a biquadratic surface with double points in the given points [1]). From this property it also follows that c is a quadruple line of O_l. Consequently O_l is a surface of the sixth degree that has a double point in T and on which c is a quadruple line.

By investigating in how many points O_l is cut outside c and T by the pair of lines a_P, a'_P and by k_b, we find the following numbers, of which the former also directly follows from a property indicated in § 5:

There are four curves k^4 that pass through a given point P and cut a given line l.

There are eight curves k^4 that have a given chord b and cut a given line l.

§ 8. Two surfaces O_l and O_m cut each other along the line c, which must be counted sixteen times, and a curve k_{lm} of the order twenty, which is obviously the image curve of the system of the curves k^4 that cut two given lines l and m. k_{lm} has a quadruple point in T and cuts c in sixteen points.

Intersection of k_{lm} with K^2 and O_n gives:

There are 32 curves k^4 that touch a_1 and cut two given lines l and m.

There are 48 curves k^4 that cut three given lines l, m and n.

§ 9. The system Σ_3 of the curves k^4 that cut a_2 twice and, accordingly, each have two image points on a line through T, is represented on a plane α_{a_2} that passes through T. For the curves of Σ_3 cut a_1 in pairs of points of the quadratic involution on a_1 produced by the pencil of quadratic surfaces that pass through $A_1, \dots, A_5$ and contain a_2.

If we cut α_{a_2} by the pair of lines a_P, a'_P and by k_b, we find the following numbers, of which the former is again an immediate consequence of a property indicated in § 5:

There is one curve k^4 that cuts a_2 twice and passes through a given point P.

There is one curve k^4 that cuts a_2 as well as a given line b twice.

§ 10. The congruence of the biquadratic curves k'^4 of the first kind that pass through seven given points $A_1, \dots, A_7$, may be brought into a one-one-correspondence with the points of the plane $\alpha \equiv A_1 A_2 A_3$ by

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[1]) Cf. Prof. JAN DE VRIES, Eine Kongruenz von Raumkurven vierter Ordnung, erster Art. Nieuw Archief v. Wisk. **15**, 229.

associating to any curve k'^4 its fourth point of intersection with α as image point. A_1, A_2 and A_3 are singular points. In A_i are represented the ∞^1 curves k'^4 that touch α in A_i and lie on the quadratic surface $\omega^2{}_i$ that passes through $A_4, \ldots, A_7$ and contains the lines $A_i\,A_k$ and $A_i\,A_l$.

A line a through A_i in α is the image of the system of the ∞^1 curves k'^4 that lie on the quadratic surface ω^2 which passes through $A_4, \ldots, A_7$ and contains the lines a and $A_k\,A_l$. ω^2 as well as $\omega^2{}_i$ contain two curves k'^4 that cut a given line l. Hence the system of the curves k'^4 that cut a given line l, is represented on a biquadratic curve k_l with double points in A_1, A_2 and A_3, which is evidently the intersection of α and the surface formed by the curves k'^4 that cut l. In this way we have proved the property applied in § 7.

ω^2 as well as ω^2_i contain six curves k'^4 that touch a given plane φ. Accordingly the system of the curves k'^4 that touch a given plane φ, is represented on a curve k_φ of the twelfth order with sextuple points in A_1, A_2 and A_3. k_φ cuts k_l outside A_1, A_2 and A_3 in twelve points.

There are, therefore, twelve curves k'^4 that cut a given line and touch a given plane.

The intersection of two curves k_φ en k_ψ also gives:

There are 36 curves k'^4 that touch two given planes.

§ 11. Through application of the above we can show in a simple way that the system of the curves k^4 that touch a given plane φ, is represented on a surface O_φ of the 18[th] degree that has a sextuple point in T and on which c is a twelve-fold line.

Intersection of O_φ with (a_P, a'_P), k_b and k_{lm} gives the following numbers, of which the former again follows immediately from § 5:

There are twelve curves k^4 that pass through a given point P and touch a given plane φ.

There are 24 curves k^4 that have a given chord b and touch a given plane φ.

There are 144 curves k^4 that cut two given lines l and m and touch a given plane φ.

§ 12. We can also investigate the representations of different other systems of curves k^4, as e.g. the systems of the curves k^4 that touch two given planes, that cut a given line and touch a given plane, and others.

The numbers that can be deduced from this and the numbers deduced above are the following ones:

$P^7\nu^2 = 4$	$P^6 T\nu = 4$	$P^5 B^3 = 1$	$P^5 BT\nu = 4$	$P^5 B\nu^4 = 48$
$P^7\nu\varrho = 12$	$P^6 B^2 = 1$	$P^5 B^2\nu^2 = 8$	$P^5 T\nu^3 = 32$	$P^5 B\nu^3\varrho = 144$
$P^7\varrho^2 = 36$	$P^6 B\nu^2 = 4$	$P^5 B^2\nu\varrho = 24$	$P^5 T\nu^2\varrho = 96$	$P^5 B\nu^2\varrho^2 = 432$
	$P^6 B\nu\varrho = 12$	$P^5 B^2\varrho^2 = 72$	$P^5 T\nu\varrho^2 = 288$	$P^5 B\nu\varrho^3 = 1296$

Here P represents the condition that a biquadratic curve of the first kind pass through a given point, ν that it cut a given line once, B that it cut a given line twice, T that it touch a given line and ϱ that it touch a given plane.

§ 13. The above enables us to indicate properties of surfaces formed by systems of ∞^1 curves k^4, such as:

The curves k^4 that have a given chord b, form a surface of the eighth degree with quadruple points in $A_1, \ldots, A_5$, on which a_1 and b are double lines and a_2 is a single line.

The curves k^4 that cut two given lines l and m, form a surface of the degree 48 with 24-fold points in $A_1, \ldots, A_5$, on which a_1 is a 16-fold line and a_2, l and m are 8-fold lines. The multiplicity of a_1 is equal to the number of curves k^4 that pass through a given point of a_1 and cut l and m. From the property indicated in § 8 that k_{lm} cuts c in 16 points, it follows that this number is equal to 16. Etc.

Mathematics. — *Determination of the Elementary Numbers of the Quadratic Surface by means of Representations of Systems of Quadratic Surfaces on the Points of a Linear Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of September 29, 1928)

§ 1. In the same way as SCHUBERT, Kalkül der abzählenden Geometrie, § 22, where the elementary numbers of the quadratic surface by means of degenerations have been determined, we indicate by μ the condition that the quadratic surface pass through a given point, by ϱ that it touch a given plane, by ν that it touch a given straight line, by φ a quadratic surface that is degenerate in a double degree-plane containing a single rank-conic which is at the same time a single class-conic and by ψ a quadratic surface which is degenerate in two single degree-planes of which the intersection is a double rank-line containing two single class-points.

In what follows three numbers are supposed to be known, viz. $\mu^9 = 1$, $\mu^8\varrho = 3$ and $\mu^8\nu = 2$.

§ 2. *The system* μ^7.

Let us suppose the two-dimensional set of the quadratic surfaces F^2 that pass through seven given points $A_1, \ldots, A_7$ to be represented projectively on the points of a plane α. This representation does not possess any singular elements. A line of α is the image of a pencil of surfaces F^2. The systems of the surfaces F^2 that touch a given plane, resp. a given line, are represented on a cubic, resp. a conic. Consequently $\mu^7\varrho^2 = 9$, $\mu^7\nu\varrho = 6$ and $\mu^7\nu^2 = 4$.

§ 3. *The system* μ^6.

We suppose the three-dimensional set of the quadratic surfaces F^2 that pass through six given points $A_1, \ldots, A_6$ to be projectively represented on the points of a linear three-dimensional space R_3.

The planes $A_1 A_2 A_3$ and $A_4 A_5 A_6$ form a pair of planes that is a part of ∞^2 degenerations ψ that are represented in one point H_{23}. Accordingly our representation has ten cardinal points H_{ik} (i, $k = 2, \ldots, 6$; $i \neq k$).

A plane in R_3 is the image of a net of surfaces F^2 and a line in R_3 is the image of a pencil of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic surface F_α that has single points in H_{ik}. We prove

the latter by cutting F_α by a line a that passes e.g. through H_{23} and is, accordingly, the image of a pencil of quadratic surfaces F^2 to which the pair of planes $A_1 A_2 A_3$, $A_4 A_5 A_6$ belongs. This pencil contains two non-degenerate surfaces F^2 that touch α. Consequently F^2 is cut by a outside H_{23} in two points.

The system of the surfaces F^2 that touch two given planes α and β, is represented on the curve of intersection $k_{\alpha\beta}$ of F_α and F_β, which is of the ninth order and has single points in H_{ik}.

$k_{\alpha\beta}$ cuts F_γ outside the cardinal points in 17 more points. Hence: $\mu^6\varrho^3 = 17$.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic cone K_l. The image surface is quadratic because a pencil of quadratic surfaces contains two individuals that touch a given straight line. Let T_l be the image of the surface F^2 that contains l. A line r through T_l is the image of a pencil of surfaces F^2 to which belongs the surface F^2 that contains l. Now as a rule this pencil does not contain any surface that touches l. Accordingly r has no point outside T_l in common with the image surface. The image surface is, therefore, a quadratic cone with vertex in T_l.

The system of the surfaces F^2 that touch two given lines l and m, is represented on the biquadratic curve of intersection k_{lm} of the surfaces F_l and F_m. Neither k_{lm} nor F_l pass through H_{ik}.

The intersection of $k_{\alpha\beta}$ and F_l, of k_{lm} and F_α and of k_{lm} and F_n gives the numbers $\mu^6\nu\varrho^2 = 18$, $\mu^6\nu^2\varrho = 12$ and $\mu^6\nu^3 = 8$.

§ 4. *The system μ^5.*

We suppose the four-dimensional system of the quadratic surfaces F^2 that pass through five given points $A_1, \ldots, A_5$, to be projectively represented on the points of a linear four-dimensional space R_4.

With any plane through $A_4 A_5$ the plane $A_1 A_2 A_3$ forms a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^3 degenerations ψ is represented on a cardinal line h_{45}. Evidently the representation has ten cardinal lines h_{ik} $(i, k = 1, \ldots, 5;\ i \neq k)$. Each of the cardinal lines, e.g. h_{46}, is cut by three of the others, i.c. h_{12}, h_{23} and h_{13}.

A linear i-dimensional space $(i = 1, 2, 3)$ in R_4 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic space Ω_α of which the lines h_{ik} are single lines. We prove the latter by cutting Ω_α by a line that cuts a line h_{ik}.

The system of the surfaces F^2 that touch two given planes α and β, is represented on the surface of intersection $F_{\alpha\beta}$ of Ω_α and Ω_β, which is of the ninth degree and of which the lines h_{ik} are single lines.

$F_{\alpha\beta}$ is cut by Ω_γ along the lines h_{ik} and a curve $k_{\alpha\beta\gamma}$ of the order 17, which is the image curve of the system of the surfaces F^2 that touch three given planes α, β and γ. $k_{\alpha\beta\gamma}$ cuts each of the lines h_{ik} in three

points. For we can show that each of the planes $A_i\ A_k\ A_l$ is a part of three degenerate surfaces F^2 that touch α, β and γ.

$k_{\alpha\beta\gamma}$ is cut by Ω_δ outside cardinal points in $3.17 - 10.3 = 21$ points. Hence: $\mu^5 \varrho^4 = 21$.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic space Ω_l with a double line d_l. d_l is the image of the pencil of surfaces F^2 that contain l. Ω_l touches the lines h_{ik}.

The system of the surfaces F^2 that touch two given lines l and m, is represented on the surface of intersection F_{lm} of Ω_l and Ω_m, which is of the fourth degree and has four double points viz. in the points of intersection of d_l and Ω_m and in those of d_m and Ω_l.

The system of the surfaces F^2 that touch three given lines l, m and n, is represented on the curve of intersection k_{lmn} of Ω_l, Ω_m and Ω_n, which is of the eighth order.

Intersection of $k_{\alpha\beta\gamma}$ and Ω_l, of $F_{\alpha\beta}$ and F_{lm}, of k_{mn} and Ω_α and of F_{lm} and F_{no} gives the numbers: $\mu^5 \nu \varrho^3 = 34$, $\mu^5 \nu^2 \varrho^2 = 36$, $\mu^5 \nu^3 \varrho = 24$, and $\mu^5 \nu^4 = 16$.

§ 5. *The system μ^4.*

We suppose the five-dimensional system of the quadratic surfaces F^2 that pass through four given points $A_1, \ldots, A_4$, to be projectively represented on the points of a linear five-dimensional space R_5.

The plane $A_1\ A_2\ A_3$ forms with any plane through A_4 a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^4 degenerations ψ is represented on a cardinal plane σ_4. Our representation has four cardinal planes σ_i $(i = 1, \ldots, 4)$ of which any two have a point in common.

In the same way any plane through $A_1\ A_2$ forms with any plane through $A_3\ A_4$ a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^4 degenerations ψ is represented on a cardinal surface σ_3^2 which is quadratic because through two arbitrary points there pass two pairs of planes of which one plane passes through $A_1 A_2$, the other through $A_3 A_4$. Any line of one scroll of σ_3^2 is the image of a system of ∞^3 degenerations ψ which have a plane through $A_1\ A_2$ in common, and any line of the other scroll of σ_3^2 is the image of a system of ∞^3 degenerations ψ that have a plane through $A_3\ A_4$ in common. The linear, three-dimensional space which contains σ_3^2, is the image of the system of the ∞^3 surfaces F^2 that contain the lines $A_1\ A_2$ and $A_3\ A_4$. Evidently there are three quadratic cardinal surfaces σ_i^2 $(i = 1, 2, 3)$. Any of the planes σ_i has a line in common with any of the surfaces σ_i^2. Two surfaces σ_i^2 have two points in common each of which is a point of intersection of two planes σ_i.

A linear i-dimensional space $(i = 1, \ldots, 4)$ in R_5 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is repre-

sented on a cubic four-dimensional space ${}^4\Omega_\alpha$ in which σ_i and σ_i^2 are single.

The system of the surfaces F^2 that touch two given planes α and β, is represented on the three-dimensional space $\Omega_{\alpha\beta}$ of the ninth degree which ${}^4\Omega_\alpha$ and ${}^4\Omega_\beta$ have in common and in which σ_i and σ_i^2 are single.

$\Omega_{\alpha\beta}$ has in common with ${}^4\Omega_\gamma$ the planes σ_i, the surfaces σ_i^2 and a surface $F_{\alpha\beta\gamma}$ of the degree 17 that is the image of the system of the surfaces F^2 that touch three given planes α, β and γ. We can prove that $F_{\alpha\beta\gamma}$ has three lines in common with any of the planes σ_i and three conics with any of the surfaces σ_i^2.

$F_{\alpha\beta\gamma}$ is cut by ${}^4\Omega_\delta$ along straight lines lying in the planes σ_i, conics lying in the surfaces σ_i^2 and a curve $k_{\alpha\beta\gamma\delta}$ of the order 21 that is the image of the system of the surfaces F^2 that touch four given planes α, β, γ and δ. $k_{\alpha\beta\gamma\delta}$ cuts each of the planes σ_i in three points and each of the surfaces σ_i^2 in ten points.

$k_{\alpha\beta\gamma\delta}$ is cut by ${}^4\Omega_\varepsilon$ outside cardinal points in $21 \,.\, 3 - 4 \,.\, 3 - 3 \,.\, 10 = 21$ points. Hence: $\mu^4\varrho^5 = 21$. This number is equal to the dual number $\mu^5\varrho^4$ already found. In the following §§ we shall derive only those numbers of which the dual numbers have not yet been determined.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic four-dimensional space ${}^4\Omega_l$ with a double plane δ_l that is the image of the system of the surfaces F^2 that contain l. ${}^4\Omega_l$ touches any of the planes σ_i along a line and any of the surfaces σ_i^2 along a conic.

The system of the surfaces F^2 that touch two given lines l and m, is represented on the biquadratic three-dimensional space Ω_{lm} that ${}^4\Omega_l$ and ${}^4\Omega_m$ have in common. Ω_{lm} contains two double conics, viz. the intersection of δ_l and ${}^4\Omega_m$, which is the image curve of the system of the surfaces F^2 that contain l and touch m, and the intersection of δ_m and ${}^4\Omega_l$, which is the image curve of the system of the surfaces F^2 that contain m and touch l. Ω_{lm} has one quadruple point of contact with any of the planes σ_i and two qnadruple points of contact with any of the surfaces σ_i^2.

The system of the surfaces F^2 that touch three given lines l, m and n, is represented on the surface of the eighth degree F_{lmn} that ${}^4\Omega_l$, ${}^4\Omega_m$ and ${}^4\Omega_n$ have in common. F_{lmn} contains twelve double points.

The system of the surfaces F^2 that touch four given lines l, m, n and o, is represented on the curve of the order sixteen k_{lmno} which ${}^4\Omega_l$, ${}^4\Omega_m$, ${}^4\Omega_n$ and ${}^4\Omega_o$ have in common.

Intersection of $k_{\alpha\beta\gamma\delta}$ and ${}^4\Omega_l$, of $F_{\alpha\beta\gamma}$ and Ω_{lm}, of $\Omega_{\alpha\beta}$ and F_{lmn}, of ${}^4\Omega_\alpha$ and k_{lmno} and of ${}^4\Omega_p$ and k_{lmno} gives the numbers:

$\mu^4\nu\varrho^4 = 42$, $\mu^4\nu^2\varrho^3 = 68$, $\mu^4\nu^3\varrho^2 = 72$, $\mu^4\nu^4\varrho = 48$ and $\mu^4\nu^5 = 32$.

§ 6. *The system* μ^3.

We suppose the six-dimensional system of the quadratic surfaces F^2 that pass through three given points A_1, A_2 and A_3, to be projectively represented on the points of a linear six-dimensional space R_6.

Together with any plane $A_1 A_2 A_3$ forms a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^5 degenerations ψ is represented on a linear three-dimensional cardinal space Σ.

Any plane through $A_1 A_2$ forms with any plane through A_3 a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^5 degenerations ψ is represented on a cubic three-dimensional cardinal space $\Sigma_3{}^3$. $\Sigma_3{}^3$ contains a system S_1 of ∞^1 planes and a system S_2 of ∞^2 lines so that each plane of S_1 and each line of S_2 cut each other. Any plane of S_1 is the image of a system of ∞^4 degenerations ψ that have a plane through $A_1 A_2$ in common and any line of S_2 is the image of a system of ∞^3 degenerations ψ that have a plane through A_3 in common. $\Sigma_3{}^3$ lies in a linear five-dimensional space which is the image of the system of the surfaces F^2 that contain the line $A_1 A_2$. There are three cubic cardinal spaces $\Sigma_i{}^3$ $(i=1, 2, 3)$. $\Sigma_i{}^3$ has in common with Σ a plane and a straight line, which is the intersection of the planes that $\Sigma_k{}^3$ and $\Sigma_l{}^3$ have in common with Σ. In the first place $\Sigma_i{}^3$ and $\Sigma_k{}^3$ have a quadratic surface in common that contains the lines which $\Sigma_i{}^3$ and $\Sigma_k{}^3$ have in common with Σ and in the second place the line that $\Sigma_l{}^3$ has in common with Σ and that does not lie on the quadratic surface. Accordingly the three cubic cardinal spaces have three lines in common.

The plane $A_1 A_2 A_3$ counted doubly is a part of ∞^5 degenerations φ that are represented in one point S which is the intersection of the common planes of the cardinal spaces $\Sigma_i{}^3$ and Σ. S is a single-point of $\Sigma_i{}^3$.

A linear i-dimensional space $(i=1, \ldots, 5)$ in R_6 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic five-dimensional space ${}^5\Omega_\alpha$ in which Σ and $\Sigma_i{}^3$ are single and that has a double point in S. We prove the latter by cutting ${}^5\Omega_\alpha$ by a line a through S that is the image of a pencil of surfaces F^2 which has a double conic through A_1, A_2 and A_3 as base curve. This pencil contains one surface that touches a given plane. Accordingly a cuts ${}^5\Omega_\alpha$ outside S in one point.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic five-dimensional space ${}^5\Omega_l$ which touches Σ along a plane and each of the spaces $\Sigma_i{}^3$ along a cubic surface, has a single point in S and contains a three-dimensional double space which is the image of the system of the planes F^2 that contain l.

The system $\mu^3 \nu^2 \varrho^3$, i.e. the system of the surfaces F^2 that touch two given lines and three given planes, is represented on a curve k^{68} that is of the order 68 as its order must be equal to the number $\mu^4 \nu^2 \varrho^3$ already found. k^{68} has a 32-fold point in S. For S is the image of the four degenerations φ that are formed by the plane $A_1 A_2 A_3$ counted doubly and the four conics in it that cut the two given lines and touch

the three given planes, which degeneration must be counted eight times [1]). We can also show that k^{68} has no point in common with Σ and that it is cut by each of the planes $\Sigma_i{}^3$ in six quadruple points.

k^{68} cuts ${}^5\Omega_l$ outside S and cardinal points in $2.68 - 1.32 = 104$ points. Hence: $\mu^3 \nu^3 \varrho^3 = 104$.

In the same way we can examine the image curves of the systems $\mu^3 \nu^3 \varrho^2$, $\mu^3 \nu^4 \varrho$ and $\mu^3 \nu^5$. By cutting these by ${}^5\Omega_l$ we find the numbers: $\mu^3 \nu^4 \varrho^2 = 112$, $\mu^3 \nu^5 \varrho = 80$ and $\mu^3 \nu^6 = 56$.

§ 7. *The system* μ^2.

We suppose the seven-dimensional system of the quadratic surfaces F^2 that pass through two given points A_1 and A_2, to be projectively represented on the points of a linear seven-dimensional space R_7.

Any plane through $A_1 A_2$ forms with any other plane a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^6 degenerations ψ is represented on a four-dimensional cardinal space ${}^4\Sigma^4$ of the fourth degree, as through four arbitrary points there pass four pairs of planes of which one plane passes through $A_1 A_2$. ${}^4\Sigma^4$ contains a system S_1 of ∞^1 linear three-dimensional spaces and a system S_2 of ∞^3 lines so that any space of S_1 cuts any line of S_2. Any space of S_1 is the image of a system of ∞^5 degenerations ψ that have a plane through $A_1 A_2$ in common. Any line of S_2 is the image of a system of ∞^3 degenerations ψ that have a plane in common, which, as a rule, does not pass through A_1 and A_2. ${}^4\Sigma^4$ lies in a linear six-dimensional space which is the image of the system of the surfaces F^2 that contain the line $A_1 A_2$.

Likewise any plane through A_1 forms with any plane through A_2 a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^6 degenerations ψ is represented on a four-dimensional cardinal space ${}^4\Sigma^6$ of the sixth degree, as through four arbitrary points there pass six pairs of planes of which one plane contains A_1 and the other A_2. ${}^4\Sigma^6$ contains two systems S'_1 and S'_2 of ∞^2 planes of which any plane of S'_1 cuts any plane of S'_2. Each plane of S'_1 or S'_2 is the image of a system of ∞^4 degenerations ψ that a plane through A_1 resp. A_2 have in common. ${}^4\Sigma^4$ and ${}^4\Sigma^6$ have two cubic three-dimensional spaces in common, which are the images of the systems of ∞^5 degenerations ψ of which one plane passes through $A_1 A_2$ and the other through A_1 resp. A_2.

Any plane through $A_1 A_2$ counted doubly is a part of ∞^5 degenerations φ. This system of ∞^6 degenerations φ is represented on a line s which the said cubic three-dimensional spaces have in common.

A linear i-dimensional space $(i = 1, \ldots, 6)$ in R_7 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic six-dimensional space ${}^6\Omega_\alpha$ in which ${}^4\Sigma^4$ and ${}^4\Sigma^6$ are single and s is a double line.

[1]) Cf. ZEUTHEN, Lehrbuch der abzählenden Geometrie, p. 351.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic six-dimensional space ${}^6\Omega_l$ that touches ${}^4\Sigma^4$ along a biquadratic three-dimensional space and ${}^4\Sigma^6$ along a three-dimensional space of the sixth degree, that has s as single line and has a linear four-dimensional double space which is the image of the system of the surfaces F^2 that contain l.

The system $\mu^2\nu^4\varrho^2$, i.e. the system of the surfaces F^2 that touch four given lines and two given planes, is represented on a curve k^{112} that is of the order 112 because its order is equal to the number $\mu^3\nu^4\varrho^2$ already found.

k^{112} is cut by s in 24 quadruple points. For there are 24 conics of which the planes pass through $A_1 A_2$ and which cut the four given lines and touch the two given planes [1]). Any of these conics together with its plane counted doubly, forms a degeneration φ that must be counted four times [2]) and is, accordingly, represented in a quadruple point of k^{112} on s. We can further show that k^{112} has no point outside s in common with ${}^4\Omega^4$ and that it is cut outside s by ${}^4\Omega^6$ in two sixteen-fold points.

k^{112} cuts ${}^6\Omega_l$ outside s and cardinal points in $2.112 - 1.96 = 128$ points. Hence: $\mu^2\nu^5\varrho^2 = 128$.

In the same way we can examine the image curves of the systems $\mu^2\nu^5\varrho$ and $\mu^2\nu^6$. By cutting these by ${}^6\Omega_l$ we find the numbers: $\mu^2\nu^6\varrho = 104$ and $\mu^2\nu^7 = 80$.

§ 8. *The system* μ.

We suppose the eight-dimensional system of the quadratic surfaces F^2 that pass through a given point A, to be projectively represented on the points of a linear eight-dimensional space R_8.

Any plane through A forms with any other plane a pair of planes that is a part of ∞^2 degenerations ψ. This system of ∞^7 degenerations ψ is represented on a five-dimensional cardinal space ${}^5\Sigma^{10}$ of the tenth degree, as through five arbitrary points there pass ten pairs of planes of which one plane passes through A. ${}^5\Sigma^{10}$ contains a system S_1 of ∞^2 linear three-dimensional spaces and a system S_2 of ∞^3 planes of which any space of S_1 cuts any plane of S_2. Any space of S_1 is the image of a system of ∞^5 degenerations ψ that have a plane through A in common. Any plane of S_2 is the image of a system of ∞^4 degenerations ψ that have a plane in common, which, as a rule, does not pass through A.

Any plane through A counted doubly is a part of ∞^5 degenerations φ. This system of ∞^7 degenerations φ is represented on a plane σ that lies in ${}^5\Sigma^{10}$.

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1) Cf. SCHUBERT, Kalkül der abzählenden Geometrie, p. 95, where the number of conics $\mu^2\nu^4\varrho^2 = 24$ is derived.

2) Cf: Note to § 6.

A linear i-dimensional space ($i = 1, \ldots, 7$) in R_8 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic seven-dimensional space ${}^7\Omega_\alpha$ in which ${}^5\Sigma^{10}$ is single and σ is a double plane.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic seven-dimensional space ${}^7\Omega_l$ that touches ${}^5\Sigma^{10}$ along a four-dimensional space of the tenth degree. σ is a single plane in ${}^7\Omega_l$ and ${}^7\Omega_l$ contains a linear five-dimensional double space that is the image of the system of the surfaces F^2 that contain l.

The system $\mu\nu^6\varrho$, i.e. the system of the surfaces F^2 that touch six given lines and a given plane, is represented on a curve k^{104}, for $\mu^2\nu^6\varrho = 104$. k^{104} is cut by σ in 52 double points and does not cut ${}^5\Sigma^{10}$ outside σ. k^{104} cuts ${}^7\Omega_l$ outside σ in $2.104 - 1.104 = 104$ points. Hence: $\mu\nu^7\varrho = 104$.

The system $\mu\nu^7$ is represented on a curve k^{80} that is cut by σ in 34 double points and that does not cut ${}^5\Sigma^{10}$. k^{80} cuts ${}^7\Omega_l$ outside σ in $2.80 - 1.68 = 92$ points. Hence: $\mu\nu^8 = 92$.

§ 9. We now suppose all quadratic surfaces F^2 of space to be projectively represented on the points of a linear nine-dimensional space R_9.

The system of the ∞^8 degenerations ψ is represented on a six-dimensional cardinal space ${}^6\Sigma^{10}$ of the degree ten, as through any six points there pass ten pairs of planes. ${}^6\Sigma^{10}$ contains a system S of ∞^3 linear three-dimensional spaces. Any space of S is the image of a system of ∞^5 degenerations ψ that have a plane in common. As a rule two spaces of S have one point in common.

The system of the ∞^8 degenerations φ is represented on a linear three-dimensional space Σ lying in ${}^6\Sigma^{10}$.

A linear i-dimensional space ($i = 1, \ldots, 8$) in R_9 is the image of a linear i-dimensional system of surfaces F^2.

The system of the surfaces F^2 that touch a given plane α, is represented on a cubic eight-dimensional space ${}^8\Omega_\alpha$ in which ${}^6\Sigma^{10}$ is single and in which Σ is a double space.

The system of the surfaces F^2 that touch a given line l, is represented on a quadratic eight-dimensional space ${}^8\Omega_l$ that touches ${}^6\Sigma^{10}$ along a five-dimensional space of the tenth degree. Σ is single in ${}^8\Omega_l$ and ${}^8\Omega$ contains a linear six-dimensional double space which is the image of the system of the surfaces F^2 that contain l.

The system ν^8 is represented on a curve k^{92} that cuts Σ in 92 points, as there are 92 conics that cut eight given lines, and that does not cut ${}^6\Sigma^{10}$ outside Σ. k^{92} cuts ${}^8\Sigma_l$ outside Σ in $2.92 - 1.92 = 92$ points.

Hence: $\nu^9 = 92$.

§ 10. The same as the systems μ^i ($i=0, \ldots, 7$) of quadratic surfaces, also the dual systems ϱ^i can be projectively represented on the points of a linear space. In stead of the degeneration φ in this case the dual degeneration χ, the quadratic cone, will play a part. By dualising the considerations of the preceding §§ we find the numbers that have not yet been derived in those §§.

Physics. — "*On a Condition for the Equilibrium of a Liquid with its Vapour Given by* BOLTZMANN, *and the Relation between this Condition and the Thermodynamic Potential.*" By J. W. DEKKER. (Communicated by Prof. J. D. VAN DER WAALS JR.).

(Communicated at the meeting of September 29, 1928).

§ 1. In § 60 of the 2nd part of his "Gastheorie" BOLTZMANN gives a condition for the equilibrium between liquid and vapour of a simple substance, which, in a somewhat more general way, may be expressed thus that the expression

$$v_b \, . \, e^{-\frac{\varepsilon_{mol}}{kT}}$$

must be the same in the two phases.

In this v_b is, per gramme molecule, the space available for a last molecule centre amidst the other molecules already present, ε_{mol} the potential energy of 1 molecule in relation to the others, kT the double of the mean kinetic energy per degree of freedom. As when n represents the number of molecules per gramme molecule, $n\,\varepsilon_{mol} = 2\,\varepsilon_{pot}$, i.e. twice the potential energy per gramme molecule, and $nkT = RT$, hence $\frac{\varepsilon_{mol}}{kT} = \frac{2\varepsilon_{pot}}{RT}$, according to the above condition

$$v_b \, . \, e^{-\frac{2\varepsilon_{pot}}{RT}}$$

hence the logarithm of this expression multiplied by $-RT$, i.e.

$$2\varepsilon_{pot} - RT \log v_b$$

must have the same value in the two phases.

I shall now show that in general this expression is not quite correct, but that

$$\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} \, . \, v - RT \log v_b \, .$$

should be substituted for it.

The former expression is in conformity with this only when $\varepsilon_{pot} = -\frac{a}{v}$, which expression BOLTZMANN also uses in the § mentioned.

§ 2. Let us first follow the course taken by BOLTZMANN.

When 1 gramme molecule of liquid coexists with 1 gramme molecule

of vapour, and we bring one more molecule inside the space in which the two phases are found, a volume $(v_b)_1$ is available in the liquid, a volume $(v_b)_2$ in the vapour for the centre of this molecule.

If there were no cohesion, the ratio between these available volumes would be at the same time the ratio of the chances that, with given position of the other molecules, the last molecule would be in the liquid or in the vapour.

BOLTZMANN multiplies the ratio of the available volumes on account of the prevailing attractive forces by

$$e^{-\frac{(\varepsilon_{mol})_1}{kT}} : e^{-\frac{(\varepsilon_{mol})_2}{kT}}$$

to arrive at the ratio of chance mentioned.

In this he interchanges, however, the potential energy ε_{mol} of some molecule in one of the two phases with the potential energy of the last molecule which we think admitted extra with given position of the other molecules. It is this latter energy that we must take into account for the ratio of probability, and in general this energy is not equal to ε_{mol}.

We have much sooner to do here with the increase of energy at constant volume (and constant temperature) when the number of molecules in the volume v is brought from n to $n+1$ than with the energy per molecule [1]).

This increase of energy, $\left(\frac{\partial \varepsilon_{pot}}{\partial n}\right)_v$, may be expressed by the aid of $\left(\frac{\partial \varepsilon_{pot}}{\partial v}\right)_n$.

For when the number of molecules is changed from n to $\varkappa n$, and at the same time the volume from v to $\varkappa v$, the potential energy becomes $\varkappa \varepsilon_{pot}$, hence it increases by $(\varkappa - 1)\varepsilon_{pot}$. For $\varkappa = 1 + d\varkappa$ this increase becomes, therefore, $d\varepsilon_{pot} = \varepsilon_{pot}\, d\varkappa$. Further dn is then $= n d\varkappa$ and $dv = v d\varkappa$. Now (T being constant) we have the following equation:

$$d\varepsilon_{pot} = \left(\frac{\partial \varepsilon_{pot}}{\partial n}\right)_v dn + \left(\frac{\partial \varepsilon_{pot}}{\partial v}\right)_n dv,$$

hence:

$$\varepsilon_{pot}\, d\varkappa = \left(\frac{\partial \varepsilon_{pot}}{\partial n}\right)_v n\, d\varkappa + \left(\frac{\partial \varepsilon_{pot}}{\partial v}\right)_n v\, d\varkappa$$

and therefore:

$$\left(\frac{\partial \varepsilon_{pot}}{\partial n}\right)_v = \frac{1}{n}\left\{\varepsilon_{pot} - \left(\frac{\partial \varepsilon_{pot}}{\partial v}\right)_n v\right\}.$$

With this value for the potential energy of the last molecule it is

[1]) I do not assign much cogency to the reasoning given in this §. I only give it to show, where and how BOLTZMANN's derivation must be modified to arrive at the correct result. More cogent proofs follow in § 3 and § 4.

found that the chances that this molecule is in one phase or in the other, are to each other as the values of the expression

$$v_b \cdot e^{-\frac{\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} v}{n k T}}.$$

The number of molecules in liquid and vapour being to each other in the same ratio as these chances, and these numbers having the same value (for we took a gramme molecule of both), the latter expression, hence also

$$\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} \cdot v - RT \log v_b$$

must be equal in the two phases.

§ 3. In a stricter way than in the preceding § we can arrive at the same result by the aid of GIBBS's statistical method, as it has been applied by ORNSTEIN to molecular problems.

Starting from the question as to which is the most frequent system divided into two homogeneous phases, from a canonical ensemble with modulus θ, ORNSTEIN finds as one of the conditions for this system, that the first derivative $f'(\mathfrak{n})$ of a certain function $f(\mathfrak{n})$ of the number of molecules $\mathfrak{n}$ per cm^3 must have the same value in the two phases. This function is:

$$f(\mathfrak{n}) = \mathfrak{n} \log \frac{\omega}{\mathfrak{n}} + \frac{a\mathfrak{n}^2}{2\theta} \text{ [1]}.$$

In this ω is in relation to our v_b in a way to be specified later; $-\frac{a\mathfrak{n}^2}{2}$ has been simply substituted for the potential energy per cm^3, hence the more general form of the function is:

$$f(\mathfrak{n}) = \mathfrak{n} \log \frac{\omega}{\mathfrak{n}} - \frac{\varepsilon_{pot}}{v\theta}.$$

It follows from this that:

$$f'(\mathfrak{n}) = \frac{d(\mathfrak{n} \log \omega)}{d\mathfrak{n}} - \log \mathfrak{n} - 1 - \frac{1}{v\theta} \cdot \frac{\partial \varepsilon_{pot}}{\partial \mathfrak{n}}.$$

Now, however, ω is in relation with v_b according to the equation:

$$\frac{d(\mathfrak{n} \log \omega)}{d\mathfrak{n}} = \log \frac{v_b}{v} \text{ [2]}.$$

while:

$$\frac{\partial \varepsilon_{pot}}{\partial \mathfrak{n}} = \frac{\partial \varepsilon_{pot}}{\partial n} \cdot \frac{n}{\mathfrak{n}} = \frac{\partial \varepsilon_{pot}}{\partial n} \cdot v = \frac{v}{n}\left(\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} \cdot v\right)$$

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[1]) Appl. de la mécanique statistique de GIBBS. Arch. Néerl. S. III A, t. 4 (1918), p. 262 et seq.

[2]) Cf. loc. cit. p. 230.

and as $-\log v - \log \mathfrak{n} = -\log n$ is the same in both cases, and $n\,\theta = RT$, our condition becomes that

$$\log v_b - \frac{1}{RT}\left(\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} \, . \, v\right),$$

hence also

$$\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} \, . \, v - RT \log v_b$$

must have the same value in the two phases.

§ 4. BOLTZMANN shows loc. cit. that, when $\varepsilon_{pot} = -\frac{a}{v}$ and $v_b = v\left(1 - 2\frac{b}{v} + \frac{17}{16}\frac{b^2}{v^2}\right)$ is put, and for the pressure the value corresponding to these values is assumed, i.e. $p = \frac{RT}{v}\left(1 + \frac{b}{v} + \frac{5}{8}\frac{b^2}{v^2}\right) - \frac{a}{v^2}$, the condition of equilibrium derived by him, after development of the logarithm to the term $\frac{b^2}{v^2}$ inclusive, assumes the same form as the so-called rule of MAXWELL: $p\,(v_2 - v_1) = \int\limits_{v_1}^{v_2} p\, dv$.

In this case the expression $2\varepsilon_{pot} - RT \log v_b$ yields the same value as the thermodynamic potential $pv - \int p\, dv$, provided no pure temperature term is introduced into the second expression.

I will now show that when the potential energy is left undetermined, and also the development according to $\frac{b}{v}$, which is questionable for the liquid phase, is omitted, the thermodynamic potential is accurately equal, with the exception of a temperature term, to the expression which I substituted for that of BOLTZMANN.

It is known that:

$$pv - \int p\, dv = \varepsilon - T\eta - \frac{\partial\,(\varepsilon - T\eta)}{\partial v} \, . \, v$$

When the kinetic energy is omitted as a pure function of the temperature, this becomes:

$$\varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} \, . \, v - T\left(\eta - \frac{\partial \eta}{\partial v} \, . \, v\right).$$

For the calculation of η we can follow the way indicated by BOLTZMANN in § 61. Hence we put:

$$\eta = Rm \log W.$$

m is the mass of a molecule (in connection with the notation adopted by us to be expressed in gramme molecules), W, the "probability of the state", consists of a temperature factor, which we may leave out of consideration here, and further of a factor indicating the number of the possibilities of distributing our n molecules over the space v, and which is put equal to the product of the volumes which the n molecules, brought each separately into the space, have at their disposal at the moment that they have been brought into it.

Hence if we represent by $2\nu m b_\nu$ the space occupied by the distance spheres of these molecules, which have a joint mass νm, when already ν molecules have been brought into the volume v, then, disregarding the temperature factor:

$$W = \prod_{\nu=0}^{n-1} (v - 2\nu m b_\nu)$$

and therefore, disregarding a temperature term:

$$\eta = R m \sum_{\nu=0}^{n-1} \log (v - 2\nu m b_\nu).$$

From this formula follows:

$$\left(\frac{\partial \eta}{\partial n}\right)_v = R m \log (v - 2 n m b_n)$$

or more simply:

$$\left(\frac{\partial \eta}{\partial n}\right)_v = R m \log v_b .$$

In an analogous way as in § 2 for the energy a relation is now derived between $\left(\frac{\partial \eta}{\partial n}\right)_v$ and $\left(\frac{\partial \eta}{\partial v}\right)_n$. It is then found that:

$$\left(\frac{\partial \eta}{\partial v}\right)_n v = R + \eta - \left(\frac{\partial \eta}{\partial n}\right)_v . n.$$ [1]

Hence:

$$\left(\frac{\partial \eta}{\partial v}\right)_n v = R + \eta - R m n \log v_b = R + \eta - R \log v_b$$

and therefore, with omission of temperature terms:

$$T\left(\eta - \frac{\partial \eta}{\partial v} . v\right) = RT \log v_b$$

and finally:

$$pv - \int p\, dv = \varepsilon_{pot} - \frac{\partial \varepsilon_{pot}}{\partial v} . v - RT \log v_b .$$

[1] The term R appears in consequence of the fact that the entropy defined in the above way is not simply multiplied by $\varkappa$ when the volume and the number of molecules are made $\varkappa$ times greater. Then the entropy becomes $\eta(\varkappa) = \varkappa . \eta + R\varkappa \log \varkappa$. With increase of $\varkappa$ from 1 to $1 + d\varkappa$ we then get $d\eta = (R + \eta)\, d\varkappa$.

Applied Mechanics. — *Temperature differences occurring in gaslift.*
By J. VERSLUYS.

(Communicated at the meeting of October 27, 1928).

Recently a theoretical analysis was given of the phenomena occurring in gaslift. (I). As gaslift is to be considered a vertical tube in which a mixture of gas and liquid rises. The pressure of the gas at the bottom of the gaslift is the main source of energy. In the above mentioned analysis it was stated that the temperature in the gaslift is not invariable, for the temperature must drop as the mixture rises. The formulae, however, were deduced as if the process were isothermal. As appears from what follows, the differences in temperature are so small, that they may be disregarded, such in view of the inevitable inaccuracies in the determination of the coefficients.

In this paper a deduction of the theory of the gaslift is given, whereby the temperature is taken as variable. It will be shown that the decrease in temperature during the rise of the gas and liquid mixture is so small that the decrease of the work exerted by the expansion of the gas in consequence of this drop in temperature may be disregarded.

The following notation will be used:

volume of liquid flowing through per unit of time	q
absolute temperature	T
absolute pressure	p
volume of gas flowing through per unit of time	$\dfrac{RT}{p}$
specific gravity of the liquid	γ_l
specific gravity of the gas at pressure p and temperature T	$\dfrac{\gamma_g p}{T}$
total volume of liquid and gas passing through the section per unit of time	v
weight of gas and liquid passing through the section per unit of time	G
ditto for gas alone	G_g
ditto for oil alone	G_l
average rate of flow of the mixture in a cross section	u
a coefficient of the dimension of rate of flow	b
specific gravity of the mixture	s
height above horizontal level of origin	y
radius of the cross section of the tube	r
amounts of energy per unit of time	W_1, W_2, etc.

mechanical heat equivalent . . . . A
specific heat of the liquid . . . . c_l
specific heat of the gas . . . . c_g
absorption coefficient . . . . α
coefficient of absorption heat . . . . β.

One may write:

$$G = q\,\gamma_l + R\gamma_g \quad \ldots \ldots \ldots \quad (1)$$

and

$$v = q + \frac{RT}{p} = \frac{qp + RT}{p} \quad \ldots \ldots \ldots \quad (2)$$

From which follows:

$$s = \frac{G}{v} = p\,\frac{q\gamma_l + R\gamma_g}{qp + RT} \quad \ldots \ldots \ldots \quad (3)$$

and

$$u = \frac{v}{\pi r^2} = \frac{qp + RT}{\pi r^2 p} \quad \ldots \ldots \ldots \quad (4)$$

If as elementary cylinder is to be understood that part of the contents of the gaslift lying between the horizontal levels at the heights y and $y + dy$, then the work of the pressure on the bottom end of this cylinder per unit of time is:

$$W_1 = pv = pq + RT \quad \ldots \ldots \ldots \quad (5)$$

from which follows:

$$dW_1 = q\,dp + R\,dT \quad \ldots \ldots \ldots \quad (6)$$

and this is the work done in the unit of time by the pressure upon the elementary cylinder.

The work performed by the force of gravity in the elementary cylinder per unit of time is:

$$dW_2 = -G\,dy = -(q\gamma_l + R\gamma_g)\,dy \quad \ldots \ldots \quad (7)$$

The mixture entering the elementary cylinder per unit of time through the bottom end would, provided the rate of flow is the same at every point in the cross section and the liquid and the gas flow at the same rate, have a kinetic energy of:

$$-\frac{G}{2g}\,u^2 = -\frac{q\gamma_l + R\gamma_g}{2g}\left(\frac{qp + RT}{\pi r^2 p}\right)^2 \quad \ldots \ldots \quad (8)$$

Assuming that the variation of this energy, caused by the speed of flow not being the same at all points of the cross section and the two substances (oil and gas) not having the same speed is expressed by a

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constant coefficient ψ, then the energy applied in this manner to the elementary cylinder per unit of time is:

$$\left.\begin{aligned} dW_3 &= -\psi \frac{q\gamma_l + R\gamma_g}{g} \frac{qp + RT}{\pi^2 r^4 p}\left(\frac{R\,dT}{p} - \frac{RT}{p^2}\,dp\right) = \\ &= -\psi \frac{(q\gamma_l + R\gamma_g)}{\pi^2 r^4 g}\left(\frac{qp + RT}{p^2}\right)\left(R\,dT - RT\frac{dp}{p}\right) \end{aligned}\right\} . \quad (9)$$

The work of the turbulence resistance may be expressed as follows, assuming that the mixture behaves as a single liquid or a single gas:

$$dW_4 = -\pi r^2 f u\,dy = -\pi r^2 f \frac{qp + RT}{\pi r^2 p}\,dy = -f\frac{qp + RT}{p}\,dy \quad . \ (10)$$

in which f (compare I p. 67) is a function:

$$f = \frac{\varphi v^2 s}{r^5} = \frac{\varphi}{r^5}\,\frac{(qp + RT)\,(q\gamma_l + R\gamma_g)}{p} \quad . \ . \ . \ . \ . \ (11)$$

φ being a constant so that:

$$dW_4 = -\frac{\varphi}{r^5}\left(\frac{qp + RT}{p}\right)^2 (q\gamma_l + R\gamma_g)\,dy \quad . \ . \ . \ . \ (12)$$

The work per unit of time performed by the expansion of the gas in the elementary cylinder is:

$$dW_5 = p\,dv = p\left(\frac{R}{p}\,dT - \frac{RT}{p^2}\,dp\right) = R\,dT - \frac{RT}{p}\,dp \quad . \ (13)$$

Finally, the energy per unit of time in the elementary cylinder converted into heat due to the difference in speed (see I page 67) may be expressed as the product of the weight of the liquid in the cylinder and a coefficient b, which, if the volume of the liquid is much greater than that of the gas, is about as great as the difference in the rate of flow, thus:

$$dW_6 = -b\gamma_l\,\pi r^2 \frac{pq}{pq + RT}\,dy \quad . \ . \ . \ . \ . \ (14)$$

An equation of equilibrium can be written as follows:

$$dW_1 + dW_2 + dW_3 + dW_4 + dW_5 + dW_6 = 0 \quad . \ . \ (15)$$

The external energy applied per unit of time to the elementary cylinder is the sum of dW_1, dW_2 and dW_3. Since the temperature of the substance flowing out of the cylinder is different from that of the

inflowing substance, a certain amount of heat is applied per unit of time. This is equivalent to the external energy, thus:

$$(q\,\gamma_l\,c_l + R\,\gamma_g\,c_g)\,dT - A\,(dW_1 + dW_2 + dW_3) = 0 \quad . \quad . \quad (16)$$

For the present no consideration is given to the value of the coefficient of the specific heat of the gas in connection with the change in pressure and volume. Later it will appear, that the terms in which this coefficient occur, may be disregarded for the present purpose.

Substituting in (15) and (16) the expressions of $dW_1, \ldots dW_6$ then two simultaneous differential equations are obtained, including p, dp, T, dT and dy.

The equations are considerably simplified if, as has been found admissible in many practical cases, one takes:

$$dW_1 = R\,dT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and:

$$dW_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Then (16) is written:

$$\frac{(q\gamma_l\,c_l + R\gamma_g\,c_g)\,dT}{A} - R\,dT + (q\gamma_l + R\gamma_g)\,dy = 0\,, \quad . \quad . \quad (19)$$

from which follows:

$$T = C - \frac{(q\gamma_l + R\gamma_g)\,A}{(q\gamma_l\,c_l + R\gamma_g\,c_g) - AR}\,y, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

in which C is the integration constant.

For (15) may now be written:

$$\left.\begin{aligned} R\,dT - (q\,\gamma_l + R\gamma_g)\,dy - \frac{\varphi}{r^5}\left(\frac{qp + RT}{p}\right)^2 (q\gamma_l + R\gamma_g)\,dy + \\ + R\,dT - \frac{RT}{p}\,dp - b\gamma_l\,\pi\,r^2\,\frac{pq}{pq + RT}\,dy = 0 \end{aligned}\right\} . \quad (21)$$

From this dy can be eliminated by substituting for the first two terms the form deduced from (19) and for the third term:

$$(q\gamma_l + R\gamma_g)\,dy = R\,dT - \frac{(q\gamma_l\,c_l + R\gamma_g\,c_g)\,dT}{A}, \quad . \quad . \quad . \quad (22)$$

which relation follows from (19). We then get the following equation:

$$\left.\begin{aligned} \frac{(q\gamma_l\,c_l + R\gamma_g\,c_g)\,dT}{A} - \frac{\varphi}{r^5}\left(\frac{qp + RT}{p}\right)^2 \left\{R - \frac{q\gamma_l\,c_l + R\gamma_g\,c_g}{A}\right\} dT + \\ + R dT - \frac{RT}{p}dp - b\gamma_l\,\pi r^2\,\frac{pq}{(pq + RT)}\,\frac{\{R - A\,(q\gamma_l c_l + R\gamma_g c_g)\}}{(q\gamma_l + R\gamma_g)}\,dT = 0 \end{aligned}\right\} . \quad (23)$$

This differential equation in p and T could be solved, so that with the aid of (20) an expression can be found of p and y. The solution will not be sought because in the equation there are still two coefficients which presumably depend upon p, viz. φ and b. Only after experimentation an opinion can be formed in this respect. In the case where the volume of the gas exceeds by far that of the liquid the equation could be further simplified by taking:

$$qp + RT = RT \quad \ldots \ldots \ldots \quad (24)$$

Then the equation (23) would be:

$$\left.\begin{aligned} &\frac{(q\gamma_l\, c_l + R\gamma_g\, c_g)\, dT}{A} - \frac{\varphi}{r^5}\left(\frac{RT}{p}\right)^2 \left\{R - \frac{q\gamma_l\, c_l + R\gamma_g\, c_g}{A}\right\} dT + \\ &+ RdT - \frac{RT}{p}\, dp - b\gamma_l\, \pi r^2 \frac{pq}{pq+RT} \frac{\{R - A\,(q\gamma_l\, c_l + R\gamma_g\, c_g)\}}{q\gamma_l + R\gamma_g}\, dT = 0 \end{aligned}\right\} \quad (25)$$

but this applies only in exceptional cases.

The main equation here is (20), from which is to be deduced the relation between the difference in temperature $T_1 - T_2$ and that of the levels $y_1 - y_2$:

$$T_2 - T_1 = \frac{(q\gamma_l + R\gamma_g)\, A}{(q\gamma_l\, c_l + R\gamma_g\, c_g) - R}\,(y_1 - y_2) \quad \ldots \quad (26)$$

Both in the numerator and in the denominator the second term is as a rule much smaller than the first, whilst also in the form between brackets in the denominator the second term is much smaller than the first. Thus the expression of the difference in temperature is approximately:

$$T_2 - T_1 = \frac{A}{c_l}\,(y_1 - y_2) \quad \ldots \ldots \ldots \quad (27)$$

If the liquid is water then c_l is approximately 1, hence:

$$T_2 - T_1 = \frac{1}{427}\,(y_1 - y_2) \quad \ldots \ldots \ldots \quad (28)$$

the temperature being expressed in degrees centigrade and the height in metres.

If the liquid is oil, for which, e.g. $c_l = \frac{1}{2}$, then is

$$T_2 - T_1 = \frac{1}{213.5}\,(y_1 - y_2) \quad \ldots \ldots \ldots \quad (29)$$

In the same manner there may be deduced from (19):

$$dT = -\frac{1}{427\, c_l}\, dy \quad \ldots \ldots \ldots \quad (30)$$

Supposing the length of a gaslift to be 650 meters, so that the tem-

perature drops 3° Celsius, while the pressure from the bottom to the top drops from 40 to 1 atmosphere, then with isothermal expansion the gas would exert an energy per unit of volume of:

$$-\int_{p=1}^{p=40} p\, dv = \int_{1}^{40} p \frac{dp}{p^2} = \int_{1}^{40} \frac{dp}{p} = log\, 40 = \pm 6. \quad . \quad . \quad . \quad (31)$$

If the temperature afterwards drops 3° at about $T = 300$, then the decrease of volume is $\frac{1}{100}$ at a pressure 1. The work then performed is $\frac{1}{100}$. Thus with isothermal expansion the work per unit of volume would be 6, of which about $\frac{1}{6}$ % would be lost through a difference in temperature of 0.01.

If, as also occurs, the pressure in the top of the gaslift is 4 atmospheres then the worth of isothermal expansion is $log\, 10 = 2.3$, from which would be deduced the product of the pressure prevailing, the volume and the decrease of volume 0.01 or 0.4 %.

In case of a gaslift where water is raised say 60 metres, while the pressure at the bottom is 4 atmospheres, then the temperature decrease will be $60 \times \frac{1}{427} = 0.14$. The energy exerted per unit of volume of gas at isothermal expansion would be $log\, 4 = 1.39$, whilst through decrease of temperature this would be reduced by $\frac{0.14}{300} = 0.0005$, or about 0.03%.

In the treatise (I) referred to in the beginning, the problem of the action of the gaslift was dealt with, the solubility of the gas being taken into consideration. In order to understand the heat phenomena in the deduction account would have to be taken of the absorption heat of the gas in the liquid. In the case of water and air there is little absorption. With petroleum and the co-existing gases the absorption coefficient α in units of volume may be $^1/_2$, which is to say that with a pressure of 1 atmosphere half a litre of gas, measured at that pressure, would dissolve in 1 litre of oil. If the pressure at the bottom of the gaslift is p_1 and at the top p_2, then the volume of gas set free during the rise, measured at 1 atmosphere is $\alpha (p_1 - p_2)$, and the weight of this volume of gas would be $\frac{\gamma_g\, \alpha (p_1 - p_2)}{T}$.

Assuming that the absorption heat β corresponds to the heat of evaporation, then this would presumably lie between 50 and 150. Then the absorbed heat per unit of volume of oil is:

$$\frac{\beta \gamma_g\, \alpha (p_1 - p_2)}{T}$$

and the resultant temperature decrease

$$\frac{\beta \gamma_g \, a \, (p_1 - p_2)}{c_l \, T},$$

for which may be taken as average about

$$\frac{100 \times \frac{1}{1000} \times \frac{1}{2}}{\frac{1}{2}} (p_1 - p_2) = \frac{1}{10} (p_1 - p_2).$$

The heat absorption capacity of the gas is disregarded. In very deep wells it might be possible that $(p_1 - p_2) = 100$, in which case the difference in temperature might be 10° Celsius. In less deep wells $p_1 - p_2$ might 36, and then the difference in temperature is 3.6° C.

In the latter case, where the difference of temperature is 3.6° C. the isothermal energy per unit of volume of gas would be 2.3, as already known, while the decrease of temperature would mean a loss of $\frac{3.6}{300} = 0.012$. This would be about $^1/_2\,^0/_0$. In the former case, where the difference in temperature is 10° C., with isothermal expansion the energy per unit of volume of gas would be 6.9 and the loss through difference in temperature $\frac{10}{300} = 0.033$, which is also about $^1/_2\,^0/_0$.

So in the theory of gaslift there is no reason for taking into account the differences in temperature.

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2. T. A. HALL: Fundamental principles of flowing wells, *The Oil and Gas Journal*, October 4, 1928, p. 30,

Physics. — *Methods and apparatus used in the Cryogenic Laboratory.* XX. *A high vacuum pump with great capacity.* By W. GAEDE and W. H. KEESOM. (Comm. N⁰. 195*a* from the Physical Laboratory at Leiden).

(Communicated at the meeting of October 27, 1928).

§ 1. *Introduction.* When KAMERLINGH ONNES [1]) in 1921 obtained the lowest temperature that has hitherto been reached he disposed, to suck off the vapours which developed from the liquid helium, of an aggregate composed of 12 glass and 3 steel LANGMUIR pumps connected in parallel, which had a total exhaust capacity of about 40 L/sec. at a pressure of 0.005 mm. of mercury.

In order to diminish the temperature still more, Prof. KAMERLINGH ONNES constructed afterwards an aggregate consisting of steel LANGMUIR pumps, which had an exhaust capacity of about double the capacity of the above mentioned. Experiments with this aggregate had to be delayed till a new aggregate of mechanical pumps with greater displacement should be installed, in order to diminish as much as possible the temperature of the helium cryostat, within which, in a space isolated by vacuum, the lowest temperature had to be reached.

In the meanwhile it seemed to one of us (W. H. K.) that it might still be possible to increase considerably the capacity of the high vacuum aggregate. He wished to dispose of a pump aggregate with an exhaust capacity for helium of 400 L/sec. at a pressure of 0,001 mm., so with ten times the capacity of that of 1921. This capacity might for instance be reached by means of diffusion pumps each with an exhaust capacity of 200 L. He/sec. by connecting three of them in parallel. He applied for this to the other of us (W. G.), who thereupon designed a construction, which was performed by E. LEYBOLD's Nachf. at Köln (1926).

This pump has been tested at Leiden and answered the expectation fully. It has at a pressure of 0.001 mm. an exhaust capacity of 270 L. He/sec.

The results of this test suggested that perhaps it would be possible to reach the desired capacity with one single pump of still greater capacity. The other of us then designed a new construction with the same outer dimensions, but with properly widened diffusion slit. The new pump, also constructed by LEYBOLD's Nachf., worked indeed as required. A short description is given in the following section.

[1]) H. KAMERLINGH ONNES. Comm. Leiden N⁰. 159, 1922.

§ 2. *The diffusion pump with exhaust capacity of 400 L. He/sec.* Fig. 1 represents the new pump [1]). It is not necessary to explain the principle of the pump because GAEDE [2]) and MOLTHAN [3]) have treated this kind of pump (mercury diffusion pump) in all particulars. The really new part of the construction is the double cooling, firstly, as always, with water from the main (inlet W_1, outlet W_2), secondly by means of a liquid cooled to $-10°$ to $-20°$ C. (inlet V_1, outlet V_2, cooling spiral *Sp*). This particular cooling in the high vacuum is necessary, in order that the mean free path of the gas-molecules to be exhausted in the stream of mercury vapour, becomes large enough in comparison with the large dimensions of the diffusion pump. The conical ringslit *R* has above a diameter of 93 mm and is there 0.5 mm wide and below it is 5 mm wide [4]). An iron screen Sch_3 prevents the mercury condensed against the cooling spiral *Sp* from falling against the ringslit cap, there to partly evaporate and to emit a cloud of vapour into the high vacuum. The iron screens Sch_1 and Sch_2 protect the connection with the backing pump against falling mercury and rising mercury

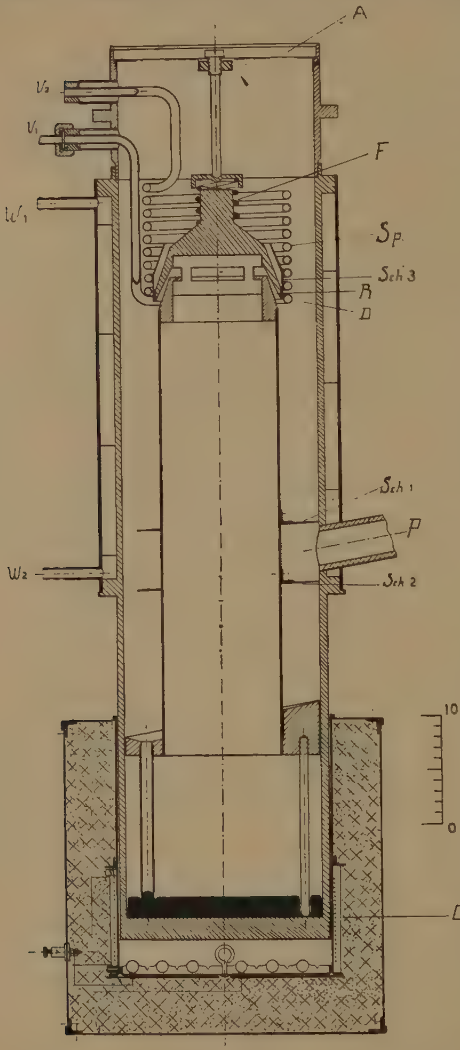

Fig. 1.

[1]) The dimensions may be deduced by means of the scale given in the figure, which gives centimeters.
[2]) W. GAEDE. Zs. f. techn. Phys. 4, 337, 1923.
[3]) W. MOLTHAN. Zs. f. techn. Phys. 7, 377 and 452, 1926.
[4]) The annular space *D* between the lowest winding of the cooling spiral *Sp* and the surrounding wall, which works as diffusion slit, has a mean diameter of about 160 mm and is about 20 mm wide.

vapour. They have a small inclination to make the condensed mercury run off. The thermal expansion of the inner part of the pump, caused by the high temperature of the mercury vapour is taken up by a steel spring *F*. At *P* the backing pump is connected. The space which is to be evacuated is connected to the pump at *A* by means of a tube with 18 cm. diameter. The pump is filled with about 7 Kg. mercury.

For the purpose of the heating an electric oven O has been constructed at Leiden. Heating may take place by two spirals of nichrome-wire; one is at the bottom and is baked in fireproof cement, resistance cold 17 Ω, the other one in the cylinder wall, resistance cold 25 Ω. This last is used at the start (15 minutes). When constant the best result is obtained by heating the bottom spiral alone, current 12.1 A (A. C. 220 V.).

§ 3. The pump was tested with air as well as with helium. For this the pump at *A* was closed with a cap (Fig. 2) in which were two holes. Through one hole the gas, which had passed a flow-meter *M* calibrated for that gas, was exhausted by the pump. The other hole served for the connection with the McLeod pressure-gauge *V*.

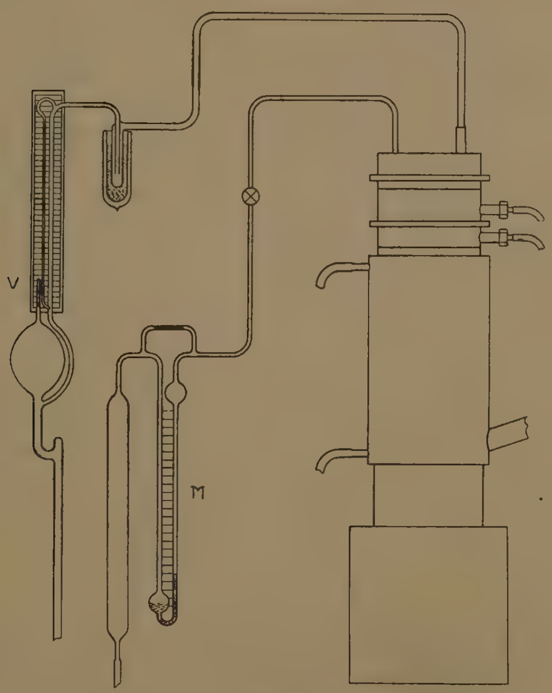

Fig. 2.

Because in these tests there was nowhere a low temperature in the space to be evacuated, it was not necessary to cool the liquid which flows through *Sp* to — 10° to — 20° C., but water was used at about 10° C.

The fore-vacuum was kept at about 0.15 mm. pressure by means of a mechanical pump with sufficient capacity.

The results are given in fig. 3. In horizontal direction is set out the pressure, measured with the McLeod, in vertical direction the displaced gas volume, measured at normal pressure. The fact that the results give straight lines, which by extrapolation pass through the origin, proves that the pump has a constant exhaust capacity to the smallest pressures at which it is tested. This exhaust capacity, that is the gas volume (measured at the pressure of the pump) which is removed per second, amounts for air to 130, for helium to 420 L/sec. With this an exhaust capacity has been obtained of fully eight times that which the 3-stage pump has at its inlet [1]).

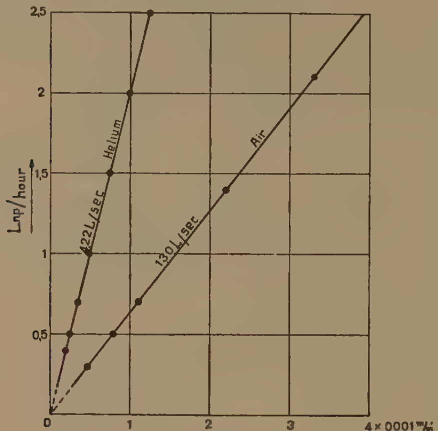

Fig. 3.

We are glad to render our thanks to G. J. Flim, chief of the technical staff of the cryogenic laboratory at Leiden, for his intelligent aid in constructing and testing this pump, and to Leybold's Nachf. for the care bestowed in executing the construction.

[1]) W. Gaede l.c. p. 368.

Microbiology. — *A new Hanseniospora.* By A. PIJPER.

(Communicated at the meeting of September 29, 1928).

The genus Hanseniospora (ZIKES), syn. Hansenia (LINDNER), has as characteristics: young cells mostly lemon-shaped, sometimes ellipsoid. Young ascospores spherical, older ascospores semi-spherical, and more or less hat-shaped. Germination by budding [1, 2]).

This genus so far comprises the species *Hansenia apiculata* (LINDNER), *Hanseniospora apiculata* (ZIKES) [3]), which forms one ascospore, and *Hanseniospora valbyensis* (KLÖCKER) [4]), which usually forms two. Recently BATSCHINSKAJA has described sixteen forms of Hanseniosporae, some of which form one, and others two ascospores [5]).

The following description applies to a third well-defined species of this genus, encountered by me in Pretoria, which forms four ascospores.

From the fingernails of a European woman, resident in Pretoria, who was suffering from onychosis, for which no cause could be found in the ordinary way, scrapings were examined microscopically and culturally. Microscopically oval "spores" were detected in the scrapings after soaking in AMANN's lactophenol and staining with methylenblue. Fifty small pieces of nailsubstance, planted out on SABOURAUD's milieu d'épreuve, in fortyfive instances gave rise to growth of a fungus identified later on as a *Hanseniospora,* whilst from the other five pieces various fungi *(Monilia, Aspergillus)* developed. As the woman was an amateur gardener, at which occupation however she always wore gloves, an examination of the soil in her garden was indicated, but from this source no *Hanseniospora* could be isolated. It must therefore be left undecided whether this *Hanseniospora* is to be regarded as pathogenic. Local and general treatment with iodine brought about slow improvement in the patient's condition.

More interest attaches to the fungus as such. Good growth occurred on all ordinary media, both at roomtemperature and in the incubator at 37°. Slightly acid media furthered growth, and beerwort was found particularly suitable. The characteristic shapes were best seen in fluid media, and are illustrated by Fig. 1. The cells were mostly lemon-shaped or ellipsoid. Occasionally, and especially in old cultures, a tendency towards formation of a simple segmented mycelium became manifest.

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1) A. GUILLIERMOND. Les Levures. Paris. 1912.
2) A. GUILLIERMOND. Clef dichotomique pour la détermination des levures. Paris. 1928.
3) ZIKES. Centralbl. f. Bakt. 1911.
4) KLÖCKER. C. R. Trav. Lab. Carlsberg. 1913.
5) BATSCHINSKAJA. Journ. microb. russe. 1926.

Multiplication, as followed under the microscope, began as a protuberance at one, or occasionally either, end of a cell. This protuberance

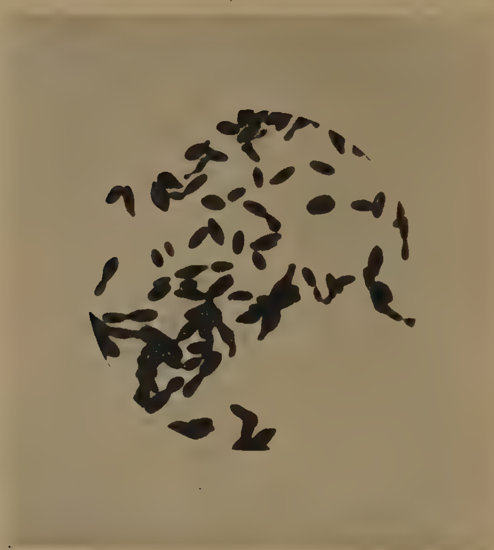

Fig. 1.
Hanseniospora guilliermondii × 1000.

remained connected with the cell by a neck for a considerable time. Then followed a constriction of the neck and the new cell at that spot became separated from the old one. As a rule the new cells became detached and took an ellipsoid or lemonlike shape soon after the constriction occurred. The comparative smallness of the cells greatly impeded morphological observations. Young cells measured about 5.2 by 2.4 microns, and dividing cells about 10.4 by 3 microns.

Production of ascospores was easily obtained when the strain was still young. The older the strain became, the more difficult it was get it to produce ascospores. In the beginning growth on blocks of plaster produced enormous numbers, but then numerous spores could also always be found on ordinary media. The number of spores in each cell was nearly always four, and their typical arrangement is shown in Fig. 2. There are two polar and two equatorial spores. The cellwall became invisible whilst the formation of spores was in progress, but the four spores remained united. There was no indication of a sexual process preceding their formation. The germination of the spores was closely followed in numerous instances, and every spore was always seen to germinate by itself, first becoming hat-shaped and then swelling up and becoming spherical, and then emitting a

tube-like protuberance which became separated from the spore and then immediately took on the characters of an ordinary vegetative cell. These

Fig. 2.
Hanseniospora guilliermondii × 1000.

phenomena were best observed, not in a hanging drop preparation, where the slightest current sweeps any particular cell out of sight, but on a solid medium. For this purpose a thin small disc of some agarmedium is sliced off the surface of a slant, and placed in the hollow of an excavated microscope slide. A trace of the material to be examined is deposited in the centre of its upper surface, and a coverglass is gently pressed on until perfect contact is obtained between coverglass and medium. The pressure distributes the cells over the surface and it was always easy to find cells which were sufficiently isolated to allow prolonged observation without interference by others. The edges of the coverglass must be wiped with vaseline.

Although from the method in which the ascospores developed it would appear as if all sexuality had disappeared, there was some evidence that the four ascospores of one cell were not always equivalent. After mordanting with chromic acid, staining with carbol-fuchsin, decolorizing with sulphuric acid, and counterstaining with methylenblue, a striking picture was obtained in some instances: the equatorial spores were acidfast, in contrast to the polar spores which showed up blue.

A difference in resistance of spores and vegetative cells could not be

demonstrated. Alcohol of 45 % killed both spores and cells within four minutes. A temperature of 57 ° was fatal in fifteen minutes for both spores and cells.

The fermentative powers of this *Hanseniospora* were very weak, as tested on the following carbohydrates : glucose, levulose, maltose, galactose, saccharose, lactose, mannite, dulcite, dextrin, raffinose, arabinose, inulin, sorbite, erythrol, glycerin and amygdalin. Acid was formed in raffinose, sorbite and dextrin, and a small quantity of gas in glucose and levulose. Beerwort was somewhat more actively fermented, but only a trace of alcohol was produced. Beerwortgelatin became liquefied after many weeks.

In all fluid media growth took place at the bottom only, no surface-film was ever formed, not even after many months.

Growth, sporulation and germination of spores took place both at room-temperature and at 37°.

Giantcolonies on beerwortagar became very large and measured six centimetres across after six months. Their colour was greyish brown, the edge lobulated and the surface smooth, showing very delicate concentric rings, corresponding in number to days of growth. There were no radial lines visible. A central knob was present from the beginning, and later on secondary similar knobs appeared at various places.

In consultation with Dr. A. GUILLIERMOND it was decided that the fungus was a *Hanseniospora*, and that, as its number of ascospores completely differentiates it from the known forms, it must be regarded as a new species. For this new species I propose the name of *Hanseniospora guilliermondii* PIJPER 1928, in honour of Dr. A. GUILLIERMOND whose work on yeasts is so universally known. I also take this opportunity of thanking Dr. GUILLIERMOND for all the help he has so kindly given me.

Pretoria, June 1928.

Bacteriology. — *Antigenic structure and specificity.* By Prof. J. J. VAN LOGHEM. (Communicated by Prof. W. SCHÜFFNER.)

(Communicated at the meeting of November 24, 1928.)

§ 1. *Introduction.*

The antibodies which appear in answer to the resorbtion of albuminoid substances fit so well into the corresponding antigens that they are used for the determination of these antigens.

The foregoing does not prove that a serological reaction is always a specific reaction. We have to consider the possibility that various species have in common the same antigen, so that experience only can show whether a serological reaction is also a specific reaction.

In pathological bacteriology especially most workers are little inclined to relinquish the great expectations, which were fostered from the serological determination of bacterial species. Others, meanwhile, recognise that the immunological reaction often fails as a specific reaction.

The first dificulty arises from bacteria, which belong to different species and cause the appearance of the same antibody. A well studied example of this kind may be found in the genus Vibrio.

Another experience which lessens the specific value of the serological reactions, one makes with bacteria of a same species which produce different antibodies. The atoxic dysentery-bacilli, the coli bacillus and the proteus-bacillus are well studied examples of this.

The third difficulty is caused by the variability of the serological results, which one may meet with in one and the same strain. Many authors do not give up the specific meaning of such results. For this reason they accept an instability of species which implies that the limits between certain bacterial species are not fixed.

Viewing the practical and theoretical signification of a sharp limiting of the notion : species, I will try to discuss the specific value of immune reactions.

§ 2. *The complex antigenic structure of bacteria.*

Already on the occasion of my first work on Proteus, I made experiments, which prove that the species Proteus anindologenes contains several antigens. The antigenic structure of many strains of this species, however, is far from complete, being only characterised by one component. So one

may meet with two strains, the specific identity of which can only be proved by the aid of a third strain.

As a scheme we may accept that there are at least two Proteus antigens : A and B. An A-strain (which contains principally the A-antigen) will give a serum, which will not agglutinate a B-strain in a typical way. So it is impossible to recognise the specific identity of strain A and strain B without the help of a strain, which contains both the antigenic components.

I will illustrate this by an example. In Amsterdam I isolated from a case of Pneumaturia a strain "Pneum", which proved to be serologically identical to a strain 22 from an absces. A few years later I isolated three Proteus anindologenes from faeces in Sumatra. Their antigenic structure, however, was different to the Amsterdam-strains. Then I received a strain from Leyde (isolated from urine). The serum, prepared by the strain-Leyde agglutinated both the Amsterdam and the Sumatra-strains.

So it is clear that the Amsterdam-strains principally contained A-antigen; the Sumatra-strains principally B-antigen ; the Leyde-strain both A- and B-antigens.

ANDREWES and others compared the complex antigenic structure of bacteria with a mosaic or a kaleidoscope ; and they have given a further analysis of this structure. The possibility of such an analysis is afforded by CASTELLANI's method. We also used this method in our work on Proteus [1]) (with BOELMAN). We proved the antigenic structure of Proteus to contain at least three components. In some strains we found all these three components, in others only 1 or 2 of them.

This comparative analysis I extended also to Proteus indologenes, which biochemically differs from Proteus anindologenes.

It was already known to us that anindologenes-sera generally do not agglutinate the indologenes-strains, or agglutinate them less well than indologenes-sera.

The positive results in a few cases meanwhile proved their relationship. The absorption tests showed that indologenes-strains and anindologenes-strains have indeed, at least, one component in common ; this means that in a given case, if this component prevails, two strains, belonging to two different species, appear as serologically identical.

At the same time it is clear that some antigens have a more "special" character, while others seem to belong more to the genus.

§ 3. *The genus-antigen prevails ; the species-antigen remains in the background.*

I have mentioned already the serological identity of different species of the genus Vibrio. Many bacteriologists do not pay any attention to the differences between Vibrio cholera and Vibrio El Tor. Yet, the points of difference between them are so important — production of an exo-toxin

[1]) Ned. Tschr. v. Geneeskunde, 1925, I. p. 1314,

and an exo-haemolysin by Vibrio El Tor [1]), where this production is negative in V. cholerae — that there is no doubt of the necessity of recognizing them as two species.

Their serological identity is only explicable by accepting in their antigenic structure a so preponderent position of the genus-components, that the species-components do not manifest themselves. Meanwhile their presence — in a latent way — is quite possible.

Often, indeed, one sees co-agglutination by cholera serum in toxic and haemolytic watervibrios. In these organisms too the genus-components remain often in the fore-ground.

§ 4. *Constant intraspecial differences in antigenic structure.*

The Proteusstrains, mentioned in § 2, illustrate not only an idea of complex antigenic structure, but they lead also to the conception that within the species difference in antigenic structure exists.

With a view to the uncertainty whether Proteus represents only one species and with a view to the variability observed in the antigenic structure in Proteus, it is preferable to study these "intraspecial differences" in species, which are less subject to variability.

Our first thoughts go to meningococcus and pneumococcus, the types of them mostly are understood as races within species. The types are different in antigenic structure; so they are examples of intraspecial difference.

On the other hand, the specific identity of these types still being in discussion, I prefer to lay stress on the fact that intraspecial differences even occur in bacterial species, about the homogeneous character of which there is no doubt at all.

A bacterium of a very constant character is the typhoidbacillus and yet, it is possible to show also in this organism an intraspecial difference in antigenic structure. It is necessary however, as I proved years ago, to use agglutinating sera of a very low titer. I will illustrate this by giving the results of an experiment. Rabbits are immunized by one injection of the strain "Ty. lab.", after one week their sera agglutinate bacilli of the same strain to 1 : 1000 and in the same dilution bacilli of the strain Str. On the other hand these sera have not the least influence on other strains of our collection neither on a very agglutinable suspension of typhoidbacilli, the diagnosticum of FICKER.

After a few weeks, during which the rabbits received several injections of the strain "Ty. lab." this "intraspecial" difference disappeared. All the above-mentioned strains and also the FICKER-suspension are now agglutinated in the same dilution.

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[1]) J. J. VAN LOGHEM, Exo-haemolysine en Endo-haemolysine bij Vibrio Tor en Vibrio cholerae, Ned. Tschr. v. Geneeskunde. 1924. II, p. 773. Bakteriophage und hämolytisches Endotoxin des Cholera vibrio. Centralbl. f. Bakt. 1926. Orig. I, Bd. **100**, p. 19.

Studying the literature at this point one does not doubt that intraspecial differences in antigenic structure are quite normal. The question whether the types of meningococcus, pneumococcus, diphtheriabacillus, Salmonella, tubercle-bacillus, tetanusbacillus, etc. are to be considered as representatives of one species is in my opinion to be confirmed.

§ 5. *Intraclonar variations in antigenic structure.*

Yet, there are differences in antigenic structure of another nature. They may appear as "intraspecial", but in reality they are variations within the clone or the individuality of the bacterium.

In accordance with an earlier publication [1]) on variations of bacteria, I remember the fact that all the bacterial cells, representing the offspring of one cell are to be considered as a clone or individual line.

It is on this individual line, that the outerworld has its effects. The influence of this outer world is partly normal and causes physiological adaptative changes of the bacteria ; it is partly abnormal and leads to an abnormal bacterial regression, characterized by a diminution, a loss or a degeneration of functions.

As an example of atrophy I mention the loss of the property to liquify gelatine or to produce spores.

An example of degeneration is the production of indol by the typhoid-bacillus.

Such variations of the clone during its development I propose to call *intraclonar variations.* They also apply to the antigenic structure.

Both the regressive and the adaptative intraclonar variations are met with intraclonar regressive variations of the antigenic structure.

As a typical example of these variations I consider the characteristic disorganisation of the antigenic structure which I have studied in old and inveterating strains of salmonella [2]).

Clones of Paratyphoid-B-strains which, immediately after their isolation, show an homogeneous structure, may after a certain time splice themselves in two individual lines of different antigenic structure.

One of these lines remains in possession of the antigenic structure of the motherstrain and contains a genus-component (co-agglutination by a typhoid-serum) ; the other line is less complex and showed a.o. the loss of the genus-component [3]).

Not long before my own work on the change in antigenic structure of inveterating Paratyphus-B-strains, others published very important findings on the variation of the antigenic structure of Proteus-strains.

I am alluding to the so well known observations of WEIL and FELIX.

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[1]) Ned. Tschr. v. Geneeskunde, 1921, II, p. 2981, and Centralbl. f. Bakt., 1922, I, Origin, Bd. **83**, p. 257

[2]) Ned. Tschr, v. Geneesk., 1919, II and Centralbl. *f.* Bakt. I, Origin., Bd. **83**.

[3]) Ned. Tdschr. v. Geneesk. 1919, II en Centralbl. f. Bakt. I Origin.. Bd. **83**.

In 1917 they obtained from complete Hauch-strains incomplete Ohne-Hauch strains the latter differ also in agglutination-tests.

The Hauch-strains possess — just as I found in the above mentioned Paratyphoid-B-strains — two antigens ; the Ohne Hauch-strains only one.

WEIL, FELIX, ANDREWES, SAVAGE, BRUCE WHITE and others examined from this stand-point first the typhoid-paratyphoid group and proved the "double type" of receptors to exist also in these bacteria.

In my opinion not all the facts, which have since been collected on the variability of the antigenic structure, may be considered as identical. The symptoms of the variability of the Paratyphoid-B-bacillus and the coli bacillus — compared with the typhoid bacillus — are partly connected with the less developed parasitic faculties of both. For the atoxic dysentery-bacillus also, we showed the adaptative changes in the antigenic structure after growth in the liquid of living tissues.

On the other hand there is an immense mass of evidence relating to variations of strains which have been examined under unfavourable conditions. Cultures of old strains, intoxicated and starved cultures form the material, in which for the last years the variability of bacteria has been studied by preference.

Agglutinative and sedimentary growth, rough and irregular colonies (change from "smooth" into "rough"), loss of virulence, loss of biochemical functions, loss of pigmentproduction, loss of capsule, spores, flagella, change of the normal shape of the bacterium cells — in short a long series of variations has been observed which — in my opinion — may be taken for regressive variations of the bacterial individuality.

The variations represent reactions on abnormal incitations from the outer-world ; they are symptoms of disease during the individual life of the bacterial clone. Sometimes they end by dying, sometimes by recovering (if the abnormal conditions are changed into normal ones), sometimes they lead to irreparable loss : to a mutilation or an atrophy, which will characterise the bacterium for the rest of its existence.

To this regressive variations — which is called dissociation by a group of American bacteriologists — I also attribute certain changes of the antigenic structure.

I think, the fact that the antigenic structure loses its genuscomponent — as was shown by me in inveterated paratyphus-B-bacilli and which has been traced also by others in the Proteusgroup as well — test explained as a fact of regression.

Although the facts of variability in bacteria are well established, the discussion as to the significance of these facts is not yet at an end. Since the publication of MASSINI on the variability of B coli it is a common practice to speak about mutants, about inheritance of acquired functions, about "Dauer-modifikation" (lasting modification) etc. In short, the variability of the bacteria is generally considered as a chapter of Genetics.

Consequently, many bacteriologists accept an instability of the bacterial species; they accept the possibility that bacteria of one species change themselves into bacteria of another species; the possibility that under the direct influence of the outer world or of the experiment, new species originate.

EISENBERG [1]) concluded his review of the bacterial variations by the words: "Eins geht aus dem vielen besprochenen Befunden zweifellos hervor, das ist die Erblichkeit erworbener Eigenschaften bei Bakterien und anderen Mikro-organismen." E. GOTSCHLICH [2]) recognises: "Tatsachen prinzipieller Bedeutung für die Ueberschreitung des Artbegriffes" and HADLEY [3]) in his otherwise so able study on Microbic Dissociation refers to "an ever increasing mass of evidence pointing to the instability of species".

In my opinion bacteriology has no right to draw conclusions on the nature and the variability of the genotypic construction of bacteria which lead to revolutionary consequences for biology.

Herewith I do not deny the possibility of real mutations — i.e. genotypic variations — in bacteria.

The words "es handele sich bei der Variabilität der Bakterien überhaupt nicht um Vererbungs- sondern nur um entwicklungsphysiologische Vorgänge" [4]) — in this way JOLLOS has formulated my view in his report in the 10th meeting of the Deutsche Vereinigung für Mikrobiologie — give an incorrect summary of my conceptions. Literally I wrote [5]): Die Erblichkeitslehre der Unizellulären kann sich nur entwickelen und ein Teil der Genetica werden, wenn sie gesäubert ist von allem, was nicht zu ihrem Objekt gehört, aber systematisch damit verwechselt wird.

b. Intraclonar adaptative variations of the antigenic structure.

In contrast with the regressive variations one may meet with changes in a bacterial individual, best understood as a normal reaction on a normal influence from the outerworld.

According to the conception given by me these variations belong to the physiology of the bacterial individuality and represent its adaptation It is especially these adaptative variations that give rise to so much misunderstanding.

The antigenic structure of many bacteria shows also the adaptative variability as a function.

Within the typhoid-coli-group there is a good opportunity to study the adaptative property of the antigenic structure. In some coli-strains one

1) WEICHARDT's Ergebnisse, 1914, p. 136.
2) Centralbl. f. Bakt., 1924, I, Orig. Bd. 93.
3) Journal of Infectious Diseases, 1927, Vol. 40.
4) Centralbl. f. Bakt., 1924, I. Orig. Bd. 93.
5) Centralbl. f. Bakt., 1922, I. Orig. Bd. 88, p. 257.

is struck by the fact that the characteristic function of this organism to adapt itself to natural conditions is represented by a very mobile structure.

I have shown [1]) that in such cases it is not possible to prepare an agglutinating serum; at the moment of the agglutination test the antigenic structure is changed in such a way, that antibody and antigen do not fit into each other.

It is also possible that a coli-strain of such a mobile structure, after having been cultivated under special conditions, changes the character of its structure. Then, the antigenic composition seems to fix, enabling one to prepare an agglutinating serum.

The adaptative change in antigenic structure may be accompanied by change in biochemical function (f.i. temporary loss of production of lactase).

We found something of this kind in the atoxic dysentery bacillus and thereby explained why so many workers tried in vain to find a serological or a biochemical classification of the dysentery-bacilli.

If one examines carefully some strains under the conditions of the laboratory it may be possible to observe, by agglutination reaction with an preserved serum, how the antigenic structure of certain strains changes its type. In some cases the biochemical functions of the strain vary at the same time; in other cases they do not.

These findings were made at first by me with KORTHOF [2]) and BOCHARDT [3]); later I repeated and extended them.

The dates on adaptative variations, studied in the laboratory, bring us to the question whether the types, (as they have been studied especially in meningococcus and pneumococcus), are to be considered as adaptative stages of the antigenic structure, which have been fixed under natural conditions.

Herewith we come also back to the intraspecial differences in antigenic structure, mentioned in § 4; differences which even may be found in a species of such a constant character as the typhoid bacillus.

In all these dates together we again see a contribution to the knowledge of variability as a function; and to the knowledge of the loss of this function in case of the increasing of the parasitic properties.

The parasitic typhoid bacillus shows an almost immovable antigenic structure and its intraspecial functions are only slightly indicated.

The organisms which are more inclined to commensalism — f.i. meningococcus, pneumococcus, atoxic dysentery bacillus, paratyphus-B-bacillus, diphtheria-bacillus — show a smaller or greater number of serological types, as the manifestation of a movability of the antigenic structures, which leads to the fact that certain antigenic components come to the foreground.

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1) Ned. Tschr. v. Geneeskunde, 1921, II, p. 1966.

2) Thesis, University Amsterdam, 1918.

3) Ned. Tschr. v. Geneeskunde, 1923, II. p. 144.

In these organisms we meet with characteristic fixations of the antigenic structure which characterise the intraspecial types of these organisms.

The greatest movability of the antigenic structure we meet with in B. coli Yet, fixation is also possible in this species; very important is the experience that the fixed coli-bacilli are often met with under pathological circumstances and that we obtain them by cultivating normal strains in the living fluids of an animal. For more particulars I refer to the thesis of TER POORTEN [1]).

§ 6. *Summary.*

Bacteria possess a complex antigenic structure.

Some of the components belong to the genus (or the family); others have a special or specific character.

Genus-antigens may prevail in such a way that a serological reaction in order to distinguish two species from each other may fail.

On the other hand there exist also differences in antigenic structure in representives of the same species. These differences may be called intraspecial differences. They are very common and it is even possible to demonstrate their existence in very homogeneous species such as the typhoid-bacillus. In other species we meet with them as serological types. which have been recognized as pneumococcus. meningococcus. diphtheria-bacillus, tubercle bacillus, tetanusbacillus, etc.

We know further the intraclonar or individual changes of the antigenic structure and we distinguish them as adaptative or regressive ones.

The adaptative variations are physiological reactions of the clone or the individuality on normal influences from the outer-world; the various atoxic dysentery-bacilli (Flexner. Y, etcetera) are examples of adaptations of antigenic structure and biochemical functions within one and the same species. The adaptative changes of the coli-bacillus. which are very often regarded as mutation, represent another example. The adaptative intra-clonar variations may be fixed for a shorter or longer time, and they may lead to the origin of fixed intra-special differences. i.e. serological types.

The regressive variations are pathological manifestations of the clone. caused by noxious (abnormal) influences from the outerworld. They occur also in the antigenic structure. Many variations in the form of cell and colony, in growth, in biochemical function and antigenic structure — which are classified as "mutations", "Dauermodifikationen", "dissociations" etc. and which are taken, from a genetical point of view. as proof of the instability of bacterial species — are better understood as degeneration and atrophy of ill-treated clones.

Institute of Hygiene, University, Amsterdam.

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[1]) Thesis, University Amsterdam, 1920.

Mathematics. — *Sur la congruence formée par les cubiques gauches ayant cinq bisécantes*, par LUCIEN GODEAUX (Liége). (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of November 24, 1928).

M. J. DE VRIES a considéré dans une note récente [1]) la congruence linéaire formée par les cubiques gauches ayant cinq bisécantes fixes et en a donné une représentation plane. Nous nous proposons d'indiquer une représentation de cette congruence par le système des cordes d'une cubique gauche; cette représentation constitue une application d'un travail que nous avons publié autrefois [2]).

1. Commençons par considérer une courbe gauche C_6, d'ordre six et de genre trois, dans un espace Σ. Rapportons projectivement les ∞^3 surfaces cubiques F passant par C_6 aux plans d'un second espace Σ'. On sait que l'on obtient ainsi une transformation birationnelle T entre les espaces Σ, Σ' [3]). Aux plans de Σ correspondent des surfaces cubiques F' passant par une courbe C'_6 d'ordre six et de genre trois; aux droites de Σ correspondent des cubiques gauches Γ' de Σ' s'appuyant en huit points sur la courbe C'_6; de même, aux droites de Σ' correspondent des cubiques gauches Γ de Σ s'appuyant en huit points sur la courbe C_6.

Cela étant rappelé, considérons une droite a de Σ ne rencontrant pas C_6 et soit Γ' la cubique gauche qui lui correspond dans Σ'. Aux bisécantes de Γ' correspondent dans Σ des cubiques gauches Γ s'appuyant en huit points sur la sextique C_6 et en deux points sur a. Ces cubiques gauches Γ forment une congruence linéaire G.

Pour obtenir la classe de la congruence G, c'est-à-dire le nombre des cubiques gauches Γ de G ayant pour bisécante une droite d ne rencontrant ni a ni C_6, observons qu'à cette droite d correspond dans Σ' une cubique gauche $\triangle'$ et qu'aux cubiques Γ répondant à la question cor-

[1]) *The Congruence of the twisted cubics that cut five given lines twice* (These Proceedings, **31**, 1928, pp. 454—458).

[2]) *Nouveaux types de congruences linéaires de cubiques gauches* (Nouvelles Annales de Mathématique, 1909, 4e série, t. 9). Citant cette note dans son ouvrage *Algèbre à deux dimensions*" (Gand, 1920), M. STUYVAERT écrit (p. 46) que, énumérant les types obtenus par ce procédé, j'en ai laissé échapper un assez remarquable. M. STUYVAERT ne donne aucun détail sur ce type remarquable, et pour cause! Le lecteur se rendra aisément compte que dans notre note, nous avons signalé tous les types généraux pouvant être obtenus par le procédé en question.

[3]) Voir par exemple CREMONA, *Sulle transformazioni razionali nello spazio* (Rend. R. Insti,uto Lombardo, 1871, et Annali di Matematica, 1871; Oeuvres complètes, t. III).

pondent les bisécantes communes des cubiques Γ', $\triangle'$. La classe de la congruence G est par suite égale à dix.

L'utilisation de la transformation T permet de poursuivre l'étude des propriétés de la congruence G. Par exemple, les cubiques gauches dégénérées de G correspondent aux bisécantes de Γ' s'appuyant sur la courbe C'_6. Si une telle droite ne passe pas par un des huit points d'appui de Γ' sur C'_6, il lui correspond une cubique Γ de G formée d'une trisécante de C_6 et d'une conique dont le plan passe par la droite a et qui s'appuie en cinq points sur C_6. Les bisécantes de Γ' s'appuyant sur C'_6 considérées forment une surface du huitième ordre passant quatre fois par Γ'; par suite les coniques dont les plans passant par a et qui s'appuient en cinq points sur C_6 forment une surface d'ordre seize passant quatre fois par la droite a et cinq fois par la courbe C_6.

A une corde de Γ' passant par un des huit points d'appui de cette courbe sur C'_6, correspond une cubique Γ de G formée d'une trisécante t de C_6 s'appuyant sur a et d'une conique s'appuyant en un point sur a, en un point sur t et en cinq points sur C_6. Le lieu de ces coniques est une surface du sixième ordre passant doublement par C_6 et par t, simplement par a.

2. Supposons maintenant que la courbe C_6 soit formée de quatre droites a_1, a_2, a_3, a_4 deux-à-deux gauches et des deux droites b_1, b_2 s'appuyant sur ces quatre droites. Alors la courbe C'_6 est également formée de quatre droites deux-à-deux gauches a'_1, a'_2, a'_3, a'_4 et des deux droites b'_1, b'_2 s'appuyant sur ces quatre droites. On peut le montrer aisément de la manière suivante: Parmi les surfaces cubiques F passant par a_1, a_2, a_3, a_4, il y en a ∞^1, formant un faisceau, contenant comme partie la quadrique R_1 lieu des droites s'appuyant sur a_2, a_3, a_4 et complétées par un plan passant par a_1. A ces surfaces cubiques correspondent dans Σ' les plans d'un faisceau d'axe a'_1. Une surface cubique passant par

$$C_6 = a_1 + a_2 + a_3 + a_4 + b_1 + b_2$$

contient, outre ces cinq dernières droites, une génératrice de R_1. A cette surface correspond dans Σ' un plan rencontrant a'_1 en un point. Par suite, aux points de a'_1 correspondent les génératrices de la quadrique R_1. On établit de même l'existence des droites a'_2, a'_3, a'_4. Cela étant, deux surfaces cubiques par a_1, a_2, a_3, a_4 ont en commun les droites b_1, b_2 et une cubique gauche; par suite à un plan de Σ correspond dans Σ' une surface cubique passant par a'_1, a'_2, a'_3, a'_4 et par b'_1, b'_2.

A une droite de Σ correspond dans Σ' une cubique gauche ayant pour bisécantes a'_1, a'_2, a'_3, a'_4 et de même, à une droite de Σ' correspond dans Σ une cubique gauche ayant pour bisécantes les droites a_1, a_2, a_3, a_4.

Aux points de la droite b'_1 (ou b'_2) correspondent les points de la droite b_1 (ou b_2) et inversement.

Considérons une cubique gauche Γ' ayant a'_1, a'_2, a'_3, a'_4 comme

bisécantes et soit a la droite qui lui correspond dans Σ, cette droite ne rencontrant pas la courbe dégénérée C_6. Aux bisécantes de Γ' correspondent dans Σ des cubiques gauches Γ ayant comme bisécantes a_1, a_2, a_3, a_4, a, et formant une congruence linéaire G.

Comme dans le premier cas, la classe de la congruence G est égale au nombre des bisécantes communes à Γ' et à une seconde cubique gauche ayant également a'_1, a'_2, a'_3, a'_4 comme bisécantes. Par suite, la classe de G est égale à six, car les droites a'_1, a'_2, a'_3, a'_4 doivent être défalquées.

Occupons-nous encore des cubiques gauches dégénérées de la congruence G. Ces cubiques correspondent à des bisécantes de Γ' s'appuyant sur l'une des droites a'_1, a'_2, a'_3, a'_4, b'_1, b'_2. Les cordes de Γ' satisfaisant à ces conditions forment dix surfaces se répartissant en deux catégories:

1°. surface lieu des cordes de Γ' s'appuyant sur b'_1 (ou sur b'_2); cette surface est du quatrième ordre, passe doublement par Γ', simplement par a'_1, a'_2, a'_3, a'_4, b'_1 (ou b'_2), mais ne passe pas par b'_2 (ou par b'_1);

2°. cône projetant la courbe Γ' d'un de ses points d'appui sur l'une des droites a'_1, a'_2, a'_3, a'_4.

A une corde d' de Γ' s'appuyant sur b'_1 correspond dans Σ une cubique de G formée de la droite b_1 et d'une conique Δ dont le plan passe par a et qui s'appuie sur a_1, a_2, a_3, a_4, b_1. La surface engendrée par les coniques Δ fait partie de la surface que T fait correspondre au lieu des bisécantes de Γ' s'appuyant sur b'_1. Cette surface est d'ordre douze et comprend comme parties les quatre quadriques lieu des droites s'appuyant sur trois des droites a_1, a_2, a_3, a_4. Le lieu des coniques Δ est donc une surface du quatrième ordre passant doualement par la droite a, simplement par les droites a_1, a_2, a_3. a_4, b_1, b_2.

Considérons maintenant une corde f' de Γ' passant par un des points d'appui A' de cette cubique sur a'_1. A cette corde correspond une courbe de la congruence G décomposée en une droite a_{11} s'appuyant sur a_1, a_2, a_3, a_4 et en une conique Φ dont le plan passe par a_1 et qui s'appuie sur a_1, a_2, a_3, a_4, a_{11}. Au cône projetant Γ' du point A', T fait correspondre une surface du sixième ordre formée de la quadrique R_1 et d'une surface du quatrième ordre passant doublement par a_1, simplement par a_1, a_2, a_3, a_4, a_{11}, et par la seconde droite a_{12} s'appuyant sur a_1, a_2, a_3, a_4. Cette dernière surface est le lieu des coniques Φ.

En résumé, les cubiques gauches dégénérées en une droite et une conique, appartenant à la congruence G, sont formées d'une droite fixe s'appuyant sur quatre des droites a_1, a_2, a_3, a_4 et de coniques dont les plans passent par la cinquième droite. Le lieu de ces coniques est une surface du quatrième ordre passant doublement par cette cinquème droite, simplement par les quatre autres et par les deux droites s'appuyant sur celles-ci.

3. La méthode qui vient d'être développée se prête particulièrement

bien à l'étude des cas limites de la congruence G, par exemple lorsque certaines des droites a_1, a_2, a_3, a_4 sont infiniment voisines. Bornons-nous à signaler celui où ces quatre droites sont infiniment voisines. Les surfaces cubiques F passent alors par deux droites incidentes a_1, b, en se raccordant le long de b et en ayant entre elles un contact du troisième ordre en tout point de a_1. Les cubiques gauches de la congruence linéaire G ont alors deux bisécantes fixes b, a_1 et des contacts du troisième ordre avec les surfaces F en chacun de leurs points d'appui sur a_1. [1])

Liége, le 1er November 1928.

[1]) La transformation T obtenue dans ce cas a été étudiée récemment par M. LAIRESSE, dans une note en cours de publication dans les Mémoires de la Société royale des Sciences de Liége.

Botany. — *Some fossil woods from Java not yet described.* By J. PH. PFEIFFER and Jhr. F. C. VAN HEURN. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of May 26, 1928).

Of late years the silicified fossil woods from the Dutch East Indies have repeatedly drawn the attention [1]) in connection with the possibility created by modern microscopic wood-research to trace the systematic relation of its original plants with the present living trees with a fairly great certainty. The method of describing the wood-structure designed by MOLL [2]) has enabled an investigator experienced in microscopic wood-research to make a correct determination from descriptions made according to that method, even though the material is not at his disposal.

Of late a striking proof of this was given by DEN BERGER in his treatise "Fossil woods from the tertiary of South-Sumatra" (1), in which he ascertained the correct botanic relation of a number of fossils described by KRÄUSEL (6).

In another treatise the same investigator pointed out how the topographic features perceptible with small magnification are especially suitable for the identifaction of woods and how with their aid the fossil woods from the familiy of the *Dipterocarpaceae* can be easily divided into genera or groups of genera.

As in this way the knowledge of tertiary and pleistocene flora of the Dutch Indies may be considerably enlarged, we thought it desirable to publish the results of the research on a collection of these fossil woods gathered by one of us.

This collection consisting of 30 specimens was collected in the private fields of Bolang, situated at about 35 kilometers West of Buitenzorg in the former residency of Batavia. The owners of the estate, the family VON KLITZING assisted in every respect.

Silicified wood abounds in that place. A great part however is corroded and breaks up into a more or less fibrous powder. Though in some of those corroded pieces some structure could be recognised, yet they were not collected, because in making collections of stones limitation is always required and moreover well-preserved material abounded. We carefully endeavoured to collect our specimens from spots which were fairly far apart, for instance from different small ravines, different brooks, etc. In

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1) See bibliography at the end of this treatise.

2) J. W. MOLL und H. H. JANSSONIUS (11) Bd. I Allgem. Teil, p. 40.

consequence of erosion fragments of one tree have been scattered over a fairly large surface and if therefore we should collect various similar samples in a few spots, it might happen, that certain woods occurred in the collection in a number out of all proportion to the real distribution. Though naturally in collecting there remains a strongly subjective factor, yet the above circumstance has been taken into account as much as possible, so that the collection, however small it may be, yet to a certain extent gives an insight into the distribution of some of the families from which specimens occur in silicified condition.

Most pieces of silicified wood we find in cultivated soil. In primeval forests they occur less frequently. Along road-sides, on washed-over slopes in *Hevea*-gardens, in brooks and small ravines and in the small dikes around the rice-fields they usually abound. Transporting the stones however is not always easy. Many of them are so large, that one cannot lift them and the smaller ones too are exceedingly heavy. With a geologist's or carpenter's hammer one can seldom remove pieces ; some even resist a heavy sledge-hammer. If we take into account the native's aversion to carrying somewhat heavy stones and his regular attempts to get rid of this, in his opinion, useless bother by losing a number of them as soon as possible, we shall understand that this collecting, too, is not seldom attended with disappointment.

For geological and petrographical territorial details we refer to the concerning literature [1]).

Of the 30 specimens one was not to be recognized on account of strong deformation and compression ; the rest was determined as follows :

a. *Dipterocarpoxylon* 7 specimens. Nos 3, 7, 11, 13, 19, 24, 29, 31.
b. *Dryobalanoxylon* 10 specimens. Nos 1, 5, 6, 21, 22, 23, 25, 26, 32, 34.
c. *Shoreoxylon* 5 specimens. Nos 4, 9 ?, 12, 27, 30.
d. *Sapindoxylon* 4 specimens. Nos 10, 20, 28, 33.
e. *Sapindopsoxylon* 1 specimen. N°. 8.
f. *Parinarioxylon* 1 specimen. N°. 2.

From this list it again appears that the family of the *Dipterocarpaceae* and especially the genera *Dipterocarpoxylon* and *Dryobalanoxylon* dominate.

No attempt was made to establish, which of the species found of this family corresponded with the species previously described by KRÄUSEL (6) and named by him and DEN BERGER (1), though in all three genera more than one species was doubtlessly present.

In this we were led by the consideration that, seeing the great similarity the anatomical structure of cognate, still living species, shows, such an attempt would be very speculative and the result that might be obtained in that way, may for the present be esteemed of secondary importance.

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[1]) See F. C. VAN HEURN (4) and (5). The illustrations given in (4) Nos 2, 3 and 4 correspond with the Nos 1, 25 and 9 of the collection discussed here.

All of the four *Sapindoxylon*-samples show such a very close resemblance, both mutually and with the *Sapindoxylon Janssonii* KRÄUSEL described by KRÄUSEL (6), that they may be considered identical with them and we may forbear giving a description.

This is not the case with the fifth petrefaction, also showing resemblance to woods from the family of the *Sapindaceae*, but deviating perceptibly from the four mentioned above. Because, as far as we know, a description of the structure of such a fossil species does not exist, it is given below.

For the same reason a detailed description of the sample indicated as *Parinarioxylon* has been given here and it has been provided with a specific name.

However before giving these descriptions, we should make a few remarks on the conclusions which may be drawn from this collection, about the probable composition of the forest in which these fossil-species grew at one time.

Much farther than ascertaining the presence of those genera, of which at present the genus *Dryobalanops* is no more found in Java, we must not go in our opinion.

It is possible, that the species found occurred in about the same numerical proportion in which they are found at present, but it may be questioned whether from these data much may be concluded concerning the share they had in the total forest stand.

In spite of this it is quite possible that future researches in the adjoining Bantam might yield more decisive figures.

If it was to be traced by counts, what numerical significance may be attached to VAN SANDICK's observation (12), "that in some spots they are, as it were, whole forests with felled trunks lying regularly side by side", and it was to appear, that they were all or for the greater part *Dipterocarpaceae* we might draw conclusions by comparing the present number of trees per hectare.

It is a striking fact, that among the 29 fossils only one specimen was found showing a really abundant growth of tyloses. In the others this formation does not occur at all or only sporadically or locally. It is true in many samples in the woodvessels structures are observed, which at first sight are very much like tyloses, but on examination in polarised light they appear to be formed by the mutual contact of crystalsphaerites.

In KRÄUSEL's photos only in *Shoreoxylon palembangense* (KRÄUSEL) d. B. (*Caesalpinoxylon palembangense* KRÄUSEL) and in *Dryobalanoxylon Tobleri* (KRÄUSEL) d. B. an abundance of tyloses are observed.

Hence we may put the question, wether this phenomenon is connected with the fact, that timbers in which tyloses are rare, are easily penetrated. It sounds plausible that those very woods which are quickly and completely permeated with water containing silicic acid, have a greater chance of being converted into well-preserved fossils, without undergoing a biological or

chemical analysis, than those rich in tyloses ; this might explain the great number of woods poor in tyloses among the silicified fossils.

How this process of silicification takes place is likely to continue an open question for a long time. It is remarkable, that in many cases the minute crystal-structure seems to have no connection with the microscopic structure of the wood, and that nearly all organic substance has totally disappeared. The loss on ignition of such pulverised, petrified wood was determined at 0.5 % ; this must be chiefly attributed to loss of moisture. What substance or substances therefore give rise to the parking in the fossil wood is likewise unknown.

The description of the species mentioned sub. *c* (sample N⁰. 8 of the collection) is as follows :

Topography. (See fig. 1.)

G r o w t h - r i n g s. Mostly poorly defined, sometimes fairly distinct ; narrow ; characterised by a period in the fibrous tissue that is denser in the late wood than in the other parts of the growth-ring.

B o u n d a r y : moderately sharp on account of the transition of the more compact late wood to the moderately compact early wood of the next growth-ring.

W o o d - v e s s e l s : Grouping : nearly all single, groups of two or three vessels are very rare ; arrangement of the wood-vessels : scattered ; number : few (average 3—4 per mm^2), sometimes a tangential zone is found, in which the vessels are somewhat rarer ; size : for the greater part wide or very wide (300—500 μ), some moderately wide (200—300 μ); Surrounding tissues : usually surrounded by medullary rays on one of the two flanks, for the rest by paratracheal parenchyma and sometimes partly by fibrous tissue.

F i b r o u s t i s s u e : moderately dense, in the late wood denser than in the rest of the zone of growth.

M e d u l l a r y - r a y s : one kind ; structure : made up of horizontal, for a considerable part short, high cells, sometimes with a row of upright cells along the edges ; width : narrow (20—30 μ), wide one row of cells, a few some what wider (to 35 μ) ; number : moderately numerous (9—10 p. mm) [1]); height : for the greater part extremely low and very low (200—800 μ), some low (to 1.4 mm).

P a r e n c h y m a. Paratracheal tissue surrounds most of the vessels as narrow, usually fairly complete layers with tangentially directed thickenings on the flanks.

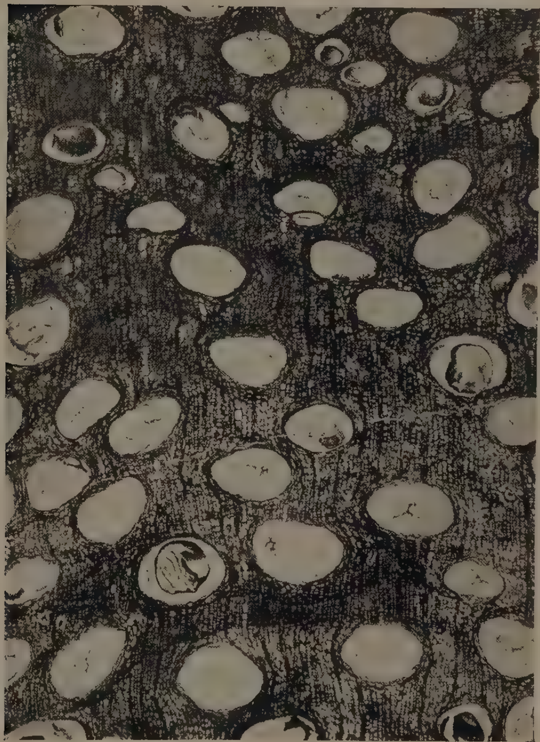
"Definitely arranged" parenchyma is absent [2]).

"Scattered" parenchyma occurs here and there as scattered strands or as small complexes.

1) In the parts slightly compressed in radial direction 8 p. mm.

2) On the meaning of the terms "definitely arranged" and "scattered" parenchyma see PH. PFEIFFER (10), pp. 37—42, 78—85, 104—107.

Sapindopsoxylon Klitzingi Pf. & v. H.



Microphoto by Dr. P. Kruizinga.

Fig. 1. Cross section, enlarged 31 times.

Special formations were not observed.

Description of the elements.

Wood-vessels. Perforations single; dimensions: in tangential direction 200—500 μ wide, in radial dirrection 150—350 μ; vascular articulations average 2 per mm. 300—600 μ long; no noticeable pitting of the walls; contents lacking, except some wood-gum [1]) here and there.

Parenchyma-cells. Dimensions: 8—25 μ thick, 20—50 μ broad, 30—130 μ long; the cells adjoining the woodvessels usually flat and short; here and there crystal-fibres are found containing short, broad cells; pitting not noticeable; contents: brown wood-gum and occasionally crystals.

Fibres. Dimensions: radially 12—35 μ, tangentially 15—30 μ, length not perceptible; contents: sometimes wood-gum.

Medullary-ray-cells. Dimensions; in radial direction 20—120 μ, in tangential direction 20—35 μ, in axial direction 15—50 μ; pitting not noticeable; contents: presumably wood-gum.

The structure of this wood usually corresponds in many respects with that of the woods from the family of the *Sapindaceae*. Its determination with the aid of a card-system arranged for this purpose by one of us, likewise led to this family. The wood however deviates considerably both from the wellknown *Sapindaceae* occurring in the Dutch East Indies, and from the *Sapindoxylon* described by KRÄUSEL (6), in one respect: that almost all the pores occur singly. Moreover the number of wood-vessels is greater than in the described *Sapindoxylon*-species, and in the other samples belonging to this genus of the collection described here.

For this reason the species described here is regarded to belong to a different genus than the *Sapindoxylon*-species and it was given the name of *Sapindopsoxylon Klitzingi* Pf. & v. H. after the owners of the private estate in which the collection was made.

It is not the intention to express with this name a certain relation between this fossil and the genus *Sapindopsis* Font., of which fossil leafrests have been found in North-America and in Europe.

The description of the structure of the fossil mentioned sub *f* (N⁰. 2 of the collection) is as follows:

Topography.

Growth-rings: as a rule wanting; here and there slightly indicated by the occurrence of a strip poor in parenchyma or by a weak period in the number of parenchyma-lines, being a little closer together in the late than in the early wood.

Boundary: where they are to be observed, vague and characterized by the sudden difference in mutual distance of the parenchyma-lines.

Woodvessels: Grouping: nearly all single; arrangement:

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[1]) Where woodgum, etc. are mentioned here we mean formations likewise consisting of silicic acid, which have wholly preserved the appearance of the substances mentioned.

scattered, only here and there an indication of complexes or more or less oblique, winding rows ; size : moderately wide and wide (200—400 μ), a very few a little wider (to 430 μ), rather variated in size, but without definite arrangement [1]); number very small (1—2 per mm^2); surrounding tissues : in a transverse section usually on both, nearly always on one of the two flanks bordered by medullary rays ; for the rest almost completely by parenchyma.

Fibrous tissue: dense and very uniform.

Medullary-rays: one kind ; structure : usually built up of horizontal cells bordered by one or two rows of high or upright cells ; here and there medullary rays with more rows of short, high cells occur, but they do not distinctly represent a different type. Width : very thin to thin, as a rule formed by one row of cells, very sporadically partly by two rows ; number : numerous, average 12 per mm ; height : for the greater part extremely low, a very few very low to low (to 1.2 mm).

Parenchyma: Paratracheal tissue present as very thin, sometimes incomplete layers, seeming to belong as it were for a great part to the metatracheal parenchyma-lines. "Definitely" arranged parenchyma occurs as numerous (average 5 per mm) very thin to thin (15—40 μ), tangentially directed, metatracheal lines, as a rule one row of cells wide, which rather undulate, change their direction or are interrupted, and locally change into definitely arranged complexes or series of scattered parenchyma. "Scattered" parenchyma only occurs here and there as scattered strands or complexes, but as a rule as rudiments of a line or series of arranged parenchyma.

Special formations: lacking, except pith-flecks observed here and there (see fig. 2).

Description of the Elements.

Woodvessels. Perforations of the vessels : single ; dimensions, radially 200—400 μ, tangentially 150—325 μ; length of the vascular articulations 450—800 μ ; average 1½ mm ; thickness of wall 3—5 μ; pits : not perceptible ; absolutely no contents.

Fibres. Dimensions : radially and tangentially 20—35 μ, mostly in tangential direction a little more stretched, length not to be observed ; thickness of wall : 5—9 μ; pits : not perceptible.

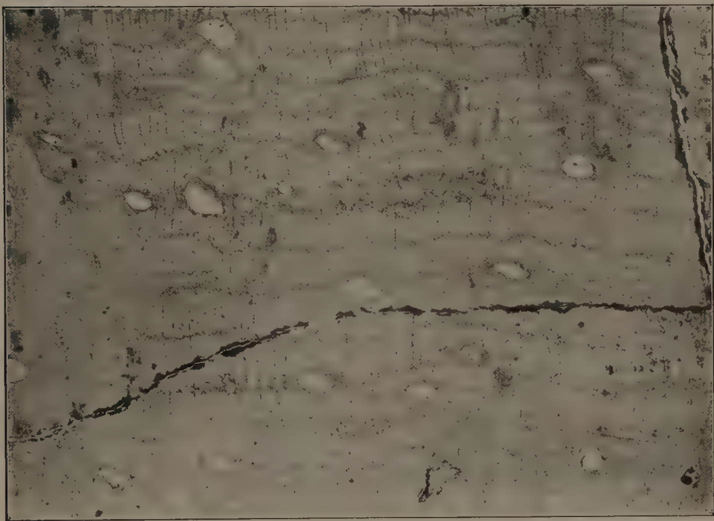
Parenchyma-cells. Dimensions radially and tangentially 20—45 μ; axially 60—100 μ; pits : not perceptible ; contents : here and there presumably wood-gum.

Medullary-ray-cells. Dimensions : radially 20—25 μ, tangentially 12—20 μ, axially 10—25 μ; pits : not perceptible contents : here and there presumably some wood-gum.

The structure of this wood shows so great a resemblance with the

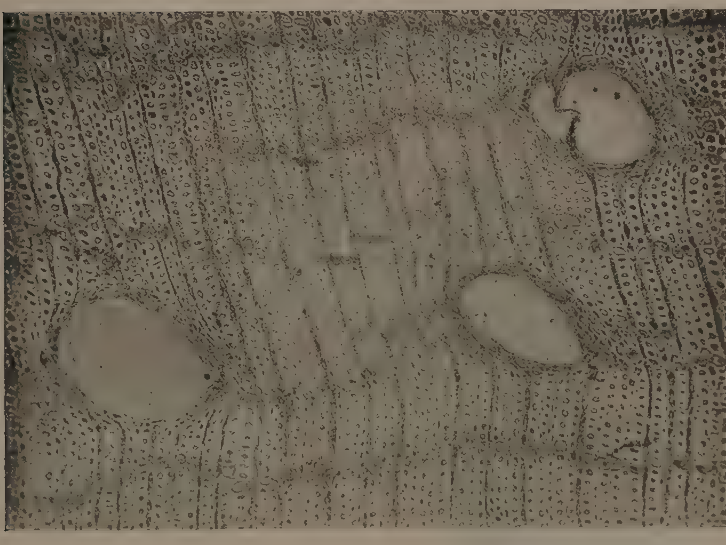
[1]) The woodvessels in the photographed part are somewhat deformed by compression in radial direction.

Parinarioxylon Itersonii Pf. & v. H.



Microphoto by Dr. P. Kruizinga.

Fig. 2. Cross section, enlarged 21 times.
In the centre a pith-ray fleck is shown.



Microphoto by Dr. P. Kruizinga.

Fig. 3. Detail from fig. 2.
Cross section, enlarged 85 times.

woods from the family of the *Rosaceae*, sub-family of the *Chrysobalanoideae*, that the relation is not at all dubious. For the rest the above description fits in details some of the *Parinarium*-species [1]) growing in the East Indies, especially *P. sumatranum* BENTH.

We therefore thought fit to give this species the name *Parinarioxylon Itersonii* Pf. & v. H. after Prof. Dr. G. VAN ITERSON Jr., through whose intermediary the laboratory of Technical Botany at Delft has been instituted.

[1]) See e.g. MOLL und JANSSONIUS (11) Vol. III, p. 222—230, DEN BERGER (2) p. 39 and J. PH. PFEIFFER (10) p. 191—195.

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